Consumer Theory

November 13, 2024

The Walrasian demand function

- When $\succsim$ is strictly convex, the solution of UPM is the Walrasian demand function, denote it $x^*(p, w)$.
- Let us now focus on $x^*(p, w)$ for some comparative-statics exercises.
- We discuss wealth effects and price effects.

- Fix the price level at $\bar{p}$, and consider $x^*(\bar{p}, w)$ as a function of w, this is the *Engel curve*.

- Consider how the demand function $x^*(\bar{p}, w)$ changes for different values of wealth, the set of all the values $\{x^*(\bar{p}, w) : w > 0\}$ is the wealth expansion path.

- Holding the price level fixed at $\bar{p}$, take $x^*(\bar{p}, w)$ differentiable. We can compute for each commodity k,

$$\frac{\partial x_k^*(\bar{p}, w)}{\partial w}$$

this is the wealth effect on the demand of good k.

- If $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} \geq 0$, good k is a normal good;
- if $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} < 0$, good k is an inferior good.
- How would the wealth expansion path of a normal good look like? and of an inferior good?

The Walrasian demand function

Comparative statics: price effects

- Starting from the Walrasian demand, consider $x^*(p, w)$ as a function of the price vector $p = (p_1, \dots, p_k, \dots, p_L)$.

- Consider the demand for commodity k, $x_k^*(p_1, \dots, p_k, \dots, p_L, w)$.

  Fix the wealth at $\bar{w}$ and the prices of all commodities except k.

  It is customary to write,
  $p = (p_k, \bar{p}_{-k})$ with $\bar{p}_{-k} = (\bar{p}_1, \dots, \bar{p}_{k-1}, \bar{p}_{k+1}, \dots, \bar{p}_L) \in \mathbb{R}^{L-1}_{++}$.

- The set of all values $\{x_k^*(p_k, \bar{p}_{-k}, \bar{w}) : p_k > 0\}$ is the *offer curve* for commodity k.

The Walrasian demand function

Comparative statics: price effects

- Let $x^*(p, w)$ be differentiable. In general,

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_k} < 0,$$

i.e. demand and price of a commodity are inversely related.

- If $\frac{\partial x_k^*(p, \bar{w})}{\partial p_k} > 0$ commodity k is a *Giffen good.*

- Think about how would the offer curve for a Giffen good look like. *[Hint: use $X = \mathbb{R}_+^2$]*

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- Think about how would the offer curve for a Giffen good look like.
  [Hint: use $X = \mathbb{R}_+^2$*]*

- We can also evaluate the effect of a change in the price of commodity j, p_j, on the demand for commodity k, $x_k^*(p, \bar{w})$, that is

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j};$$

- if commodity j is a complement for commodity k, the cross-price effect will be negative

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j} < 0;$$

- if commodity j is a substitute for commodity k, the cross-price effect will be positive

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Comparative statics and homogeneity of degree zero

- Consider property i) of Theorem 3, for the Walrasian demand function: $x^*(\alpha p, \alpha w) - x^*(p, w) = 0$. For each commodity j

$$x_j^*(\alpha p_1, \dots, \alpha p_L, \alpha w) - x_j^*(p_1, \dots, p_L, w) = 0$$

- Differentiate it w.r.t. α and evaluate at $\alpha = 1$. We get the following result.

Homogeneity of degree zero of the Walrasian demand implies that for all p and w,

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p, w)}{\partial p_k} p_k + \frac{\partial x_j^*(p, w)}{\partial w} w = 0 \qquad \text{for } j = 1, \dots, L \tag{3}$$

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Comparative statics and homogeneity of degree zero

- Take equation (3) and commodity j, divide each addend by $x_j^*(p,w)$, we get
$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} \frac{p_k}{x_j^*(p,w)} + \frac{\partial x_j^*(p,w)}{\partial w} \frac{w}{x_j^*(p,w)} = 0$$

- Recall that $\epsilon_{jk} = \frac{\partial x_j^*(p,w)}{\partial p_k} \frac{p_k}{x_j^*(p,w)}$ is the elasticity of the demand for commodity j to the price of commodity k,

  and $\epsilon_{jw} = \frac{\partial x_j^*(p,w)}{\partial w} \frac{w}{x_j^*(p,w)}$ is the wealth elasticity of the demand for commodity j.

Comparative statics and homogeneity of degree zero

- Then, equation (3)

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} p_k + \frac{\partial x_j^*(p,w)}{\partial w} w = 0$$

rewritten in terms of price and wealth elasticities, yields

$$\sum_{k=1}^{L} \epsilon_{jk} + \epsilon_{jw} = 0 \qquad \text{for } j = 1, \ldots, L$$

- When all prices and wealth change by an equal percentage, this leads to no change in demand of commodity j.

Comparative statics and Walras' law

- By Walras law, the Walrasian demand is such that $p \cdot x^*(p,w) = w$, which rewrites as

$$p_1 x_1^*(p,w) + p_2 x_2^*(p,w) + \cdots + p_L x_L^*(p,w) = w. \qquad (4)$$

a.) For each commodity j evaluate the differential change in (4) due to a change in $(p_1, ...p_L)$.

By Walras' law, for all p and w, for each commodity j,

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} p_j + x_j^*(p,w) = 0 \qquad \text{for } j = 1, \dots, L$$

For each commodity, the effect of a change in prices on its expenditure must be zero. Overall, the total expenditure cannot change if prices change.

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b.) For each commodity evaluate the differential change in (4) w.r.t. w.

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The Walrasian demand and the Weak Axiom

Lemma

The Walrasian demand function $x(p,w)$ satisfies the following property: for every two pairs (p,w), (p',w')

$$\text{if } p\cdot x(p',w')\leq w \text{ and } x(p',w')\neq x(p,w) \qquad \text{then } p'\cdot x(p,w)>w'.$$

This is a property of consistency in the choices of the consumer. Indeed, since both $x(p,w)$ and $x(p',w')$ solve UMP at the respective prices and wealth; if $x(p',w')$ is feasible at (p,w) but is not chosen and the two bundles are different, then it has to be that $x(p,w)$ is not be feasible at (p',w')...

...otherwise, one would expect the consumer to keep preferring $x(p,w)$ over $x(p',w')$ also at price (p',w')... differently, the consumer would have an inconsistent demand behavior!!

This property is the weak axiom of revealed preferences (WARP).

Implications of WARP on the price effects of the Walrasian demand

- A price change alters the relative cost of a commodity w.r.t. the other commodities in the UMP. (Substitution effect).
- Consider a change in prices accompanied by the (specific) change in w that maintains the initial consumption bundle, just affordable at the new prices.
- Start with (p, w) and the consumer optimal choice $x(p, w)$. Consider a price change to $p' \neq p$, and the change in wealth s.t. $w' = p' \cdot x(p, w)$.
- Then, $\Delta w \equiv w' - w = (p' - p) \cdot x(p, w) \rightarrow$ this is the Slutsky wealth compensation.
- The price changes accompanied by such wealth compensation are labelled as (Slutsky) compensated price changes.

WARP and the law of demand in UMP

Proposition (WARP)

The Walrasian demand function $x(p, w)$ satisfies WARP if and only if, for any compensated price change from (p, w) to $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

with strict inequality whenever $x(p', w') \neq x(p, w)$.

- The Weak Axiom imposes not only a certain consistency on the Walrasian demand, but also a form of the law of demand, in that the change in prices and in Walrasian demands move in opposite directions... at least for compensated price changes!!

- Read the differential version of Proposition (WARP) - MWG chapter 2, p. 33-34. We shall discuss it later in class.

The indirect utility function $v(p, w)$

Theorem 4

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the indirect utility function has the following properties:

i) $v(p, w)$ is homogeneous of degree zero in (p, w);
ii) $v(p, w)$ is strictly increasing in w and non-increasing in p;
iii) $v(p, w)$ is quasi-convex, that is the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$;
iv) $v(p, w)$ is continuous in p and w.

Let us prove properties $i) - ii)$.

i) Follows immediately from the fact that $x(p, w)$ is homogeneous of degree zero in (p, w).

   Indeed since $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$, then also $u(x(\alpha p, \alpha w)) = u(x(p, w))$.

ii) We prove that $v(p, w)$ is increasing in w.

Take $w' > w$, then $\mathcal{B}(p, w) \subseteq \mathcal{B}(p, w')$.

In particular, if x^* is the optimal bundle at wealth w, then x^* is feasible when wealth is w'. Hence, $v(p, w') \geq u(x^*) = v(p, w)$.

Since $\succsim$ is LNS, there exists x' that $||x' - x^*|| \leq \epsilon$ such that $x' \succ x^*$ and $x' \in \mathcal{B}(p, w')$ when ϵ is small enough, hence $u(x') > u(x^*)$.

Thus, $v(p, w') \geq u(x') > u(x^*) = v(p, w)$.

Using a similar reasoning, show that $v(p, w)$ is non-increasing in p.

The Expenditure Minimization Problem