THE UTILITY MAXIMIZATION PROBLEM

16/11/202

$$\text{Max } u(x)$$
$$\text{s.t. } \quad p \cdot x \leq w$$
$$x \geq 0$$

with L commodities

$$\text{Max } u(x_1, x_2, \ldots x_L)$$
$$\text{s.t. } \quad p_1 x_1 + p_2 x_2 + \ldots + p_L x_L \leq w$$
$$x_\ell \geq 0 \qquad \forall\, \ell = 1, \ldots, L$$

Take $L = 2$

A SUMMARY ON CONSTRAINED OPTIMIZATION

$$\text{Max } u(x_1, x_2)$$

$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq \omega$$

$$x_1 \geqslant 0$$

$$x_2 \geqslant 0$$

Examine interior solutions

Set-up the Lagrangian function

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda \left[w - p_1 x_1 - p_2 x_2 \right]$$

then find the stationary point of $\mathcal{L}(\cdot)$ by solving

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 0 \qquad x_1 > 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 0 \qquad x_2 > 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = 0 \qquad \lambda \geqslant 0 \qquad (3)$$

these FOC are necessary for an optimum.

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0 \iff \frac{\partial u(x_1, x_2)}{\partial x_1} = \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 0 \iff \frac{\partial u(x_1, x_2)}{\partial x_1} = \lambda p_2$$

↳ in matrix notation

$$\nabla u(x_1, x_2) = \lambda p \qquad x > 0$$

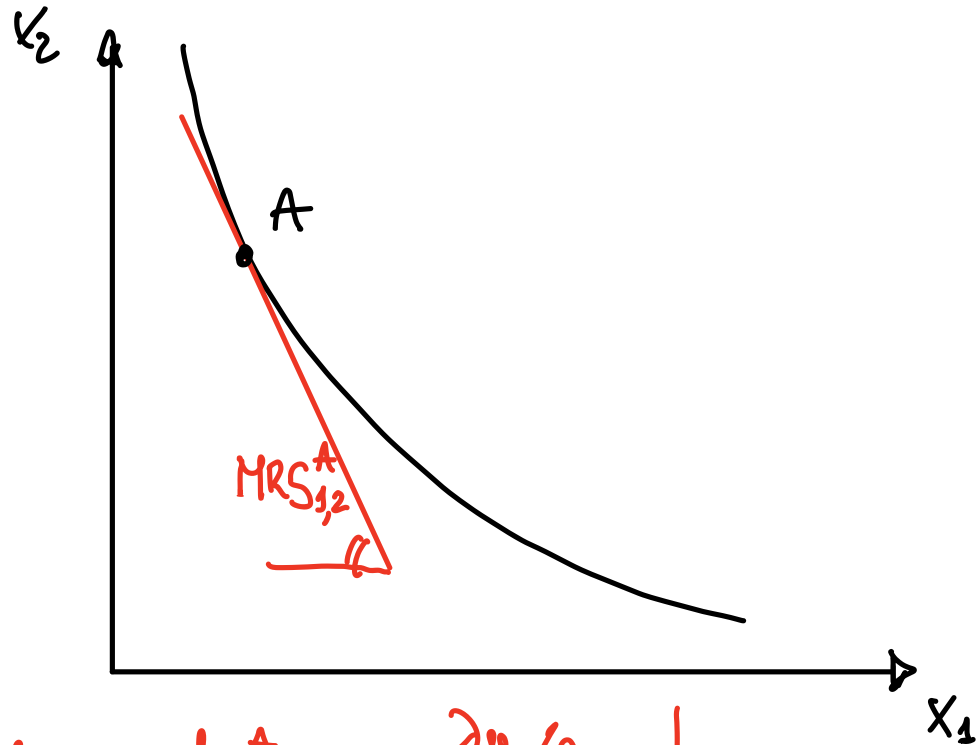
with

$$\underset{(1 \times L)}{\nabla u(x_1, x_2)} = \begin{bmatrix} \frac{\partial u(x_1, x_2)}{\partial x_1} & \frac{\partial u(x_1, x_2)}{\partial x_2} \end{bmatrix}$$

$$p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$$

the gradient vector of $u(x_1, x_2)$ and the price vector p are proportional at the optimum

CHARACTERIZATION OF AN INTERIOR SOLUTION (by NECESSARY CONDITIONS)



$$MRS_{12}^A = \frac{dx_2^A}{dx_1^A} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2}\Bigg|_A$$

measures the variation in x_2 induced by a differential change in x_1 along an indifference curve.

At the optimum $MRS_{12} = \frac{p_1}{p_2}$, why?

Assume instead: $MRS_{12} > \frac{P_1}{P_2}$

$$\Leftrightarrow \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} > \frac{P_1}{P_2}$$

Can any such bundle be optimal?

No. If this is the case the consumer can increase the consumption of good 1 by dx_1 and reduce the consumption of good 2 by $-\frac{P_1}{P_2} dx_1$ (moving along the budget line) and increase her utility by

$$\frac{\partial u}{\partial x_1} \cdot dx_1 + \frac{\partial u}{\partial x_2} dx_2 =$$

$$\frac{\partial u}{\partial x_1} dx_1 - \frac{\partial u}{\partial x_2} \cdot \left(\frac{P_1}{P_2}\right) dx_1 > 0$$

Example

Let $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$ with $\alpha + \beta = 1$
$\alpha \in (0,1)$
$\beta \in (0,1)$

Set up the UMP and find the Walrasian demand for commodity 1 and 2.

UMP $\quad \max \; x_1^{\alpha} x_2^{(1-\alpha)}$

$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq w$$

$$[x_1 \geq 0 \;,\; x_2 \geq 0]$$

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} + \lambda [w - p_1 x_1 - p_2 x_2]$$

In so doing, we are searching for interior solutions, i.e. $x_\ell^* > 0 \quad \ell = 1,2$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \qquad \lambda \geq 0$$

$$w = p_1 x_1 + p_2 x_2 \qquad (3)$$

Focus on (1)-(2)

(1) $$\alpha x_1^{\alpha-1} x_2^{1-\alpha} = \lambda p_1$$

$$\alpha \frac{x_2}{x_1} x_1^{\alpha} x_2^{-\alpha} = \lambda p_1$$

(2) $$(1-\alpha) x_1^{\alpha} x_2^{-\alpha} = \lambda p_2$$

take the ratio $\frac{(1)}{(2)}$

$$\frac{\alpha \frac{x_2}{x_1} \cancel{x_1^{\alpha} x_2^{1-\alpha}}}{(1-\alpha) \cancel{x_1^{\alpha} x_2^{-\alpha}}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2} \Rightarrow x_2 = \frac{p_1}{p_2} \cdot \frac{(1-\alpha)}{\alpha} x_1$$

substitute into (3) to get

$$p_1 x_1 + \cancel{p_2}\left(\frac{p_1}{\cancel{p_2}} \frac{(1-\alpha)}{\alpha} x_1 \right) = w$$

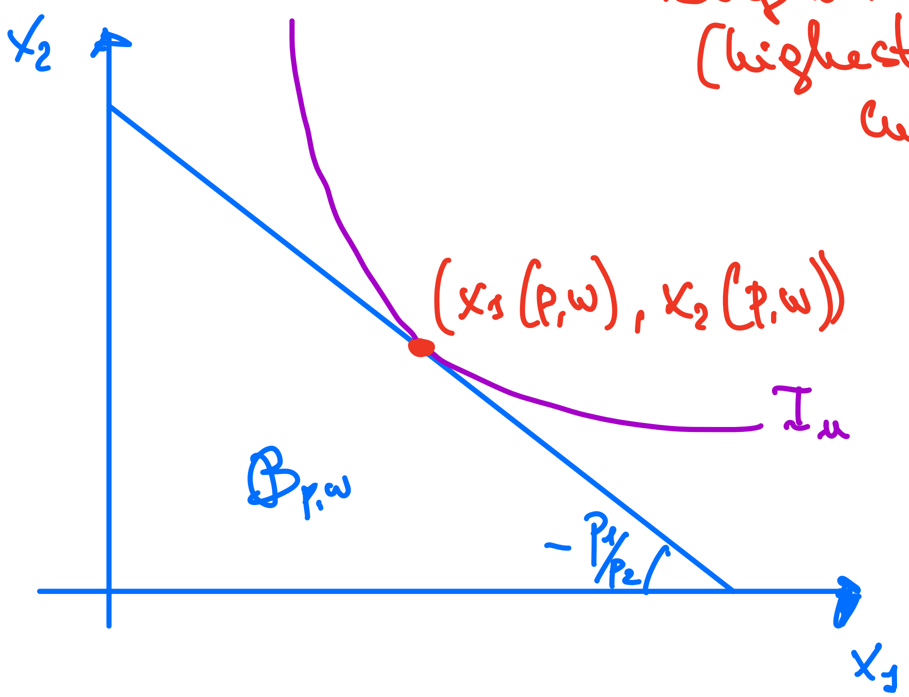
and then

$$x_1 = \frac{\alpha w}{p_1}$$

$$x_2 = \frac{(1-\alpha) w}{p_2}$$

the Walrasian demands

At the optimal bundle, the budget line is tangent to the (highest) indif. curve



ECONOMIC INTERPRETATION OF THE LAGRANGE MULTIPLIER

λ measures the mg. effect of changes in w on the UMP

λ is the mg. utility of wealth at the optimum

take $x(p,w)$ differentiable and interior solution.

$u(x(p,w))$ is the utility at the optimum

What's the effect of a change in w on $u(x(p,w))$?

$$\frac{\partial u(x(p,w))}{\partial w} = \frac{\partial u(x(\cdot))}{\partial x_1} \cdot \frac{\partial x_1(p,w)}{\partial w} + \frac{\partial u(x(\cdot))}{\partial x_2} \cdot \frac{\partial x_2(p,w)}{\partial w} + \dots + \frac{\partial u(x(\cdot))}{\partial x_L} \cdot \frac{\partial x_L(p,w)}{\partial w}$$

recall $D_w x(p,w)$ is $(L \times 1)$ vector

Hence the change in utility can be rewritten as

$$\nabla u(x(p,w)) \cdot D_w x(p,w) \qquad (C^*)$$

At the optimum

$$\nabla u(x(p,w)) = \lambda p$$

Hence (C^*) can be rewritten as

$$\lambda p \cdot D_w x(p,w) \qquad (C^*)$$

At the optimum

$$p \cdot x(p,w) = w$$

hence

$$p \cdot D_w x(p,w) = 1 \qquad (*)$$

Combining $(*)$ with (C^*) we get

$$\frac{\partial u(x(p,w)}{\partial w} = \lambda$$

the Lagrange multiplier corresponds to the mg utility of wealth; how tight is the budget constraint.

MORE ON CONSTRAINED OPTIMISATION

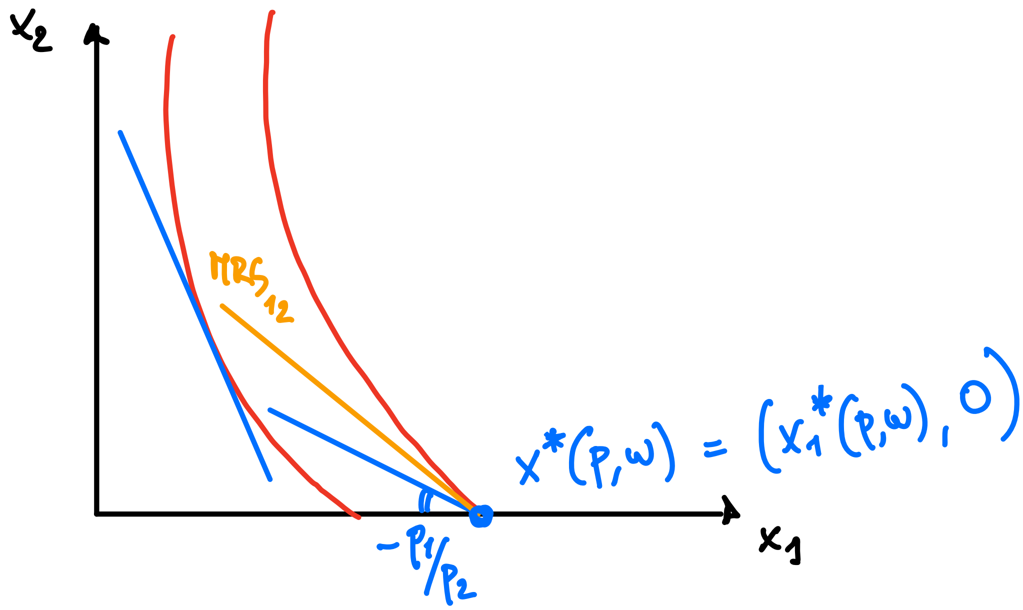
More generally, since optimal bundles may not exclude that $x_\ell^*(p,w) = 0$ for some commodity ℓ, the conditions for an optimum have to be generalized to :

$$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} \leq 0 \qquad x_1 \geq 0 \qquad x_1\left[\frac{\partial \mathcal{L}}{\partial x_1}\right] = 0 \quad (1')$$

$$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} \leq 0 \qquad x_2 \geq 0 \qquad x_2\left[\frac{\partial \mathcal{L}}{\partial x_2}\right] = 0 \quad (2')$$

$$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} \geq 0 \qquad \lambda \geq 0 \qquad \lambda\left[\frac{\partial \mathcal{L}}{\partial \lambda}\right] = 0 \quad (3')$$

IN CASE OF CORNER SOLUTIONS



At this optimum, $x_2^*(p,w) = 0$

the tangency condition here cannot hold!

$$MRS_{1,2} \neq \frac{p_1}{p_2}$$

<u>FOC</u>

$$\frac{\partial u_1(x_1,x_2)}{\partial x_1} = \lambda p_1 \quad \text{for } x_1 > 0$$

$$\frac{\partial u_2(x_1,x_2)}{\partial x_2} \leq \lambda p_2 \quad \text{for } x_2 = 0$$

Hence, the consumer would like to

reduce the consumption of x_2 as much as she can (to increase $\partial u/\partial x_2$) ... but the lower bound on consumption is $x_2 = 0$.

Hence, at the optimum for a consumer with those preferences

$$MRS_{12} > \frac{p_1}{p_2}$$

In case of corner solutions, the non-negativity constraints $x_1 \geq 0$, $x_2 \geq 0$ become relevant for optimization and should be explicitely included in the Lagrangian function, each one with its own multiplier.

It's convenient to write them as
$$\boxed{\begin{array}{l} -x_1 \leq 0 \\ -x_2 \leq 0 \end{array}}$$

so that the Lagrangian function is:

$$\mathcal{L}(x_1, x_2, \lambda, \mu_1, \mu_2) = u(x) + \lambda\left[W - p_1 x_1 - p_2 x_2\right] -$$

$$+ \mu_1 x_1 + \mu_2 x_2$$

depends on x_1, x_2 and all the multipliers λ, μ_1, μ_2.

the FOC are of the form:

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_\ell} \leq 0 \qquad x_\ell \geq 0 \qquad x_\ell\left[\frac{\partial \mathcal{L}}{\partial x_\ell}\right] = 0 \quad \forall \ell$$

and for every multiplier:

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} \geq 0 \qquad \lambda \geq 0 \qquad \lambda\left[\frac{\partial \mathcal{L}}{\partial \lambda}\right] = 0$$

IMPORTANT REMARK

the FOCs are necessary and sufficient for an optimal bundle if the utility function is strictly quasi-concave ($\succsim$ are strictly convex)

$\Rightarrow$ the bundle obtained as a result of the system of FOCs is the unique maximizer of the UMP.

Recall that

$$\begin{aligned} \text{Min} \quad & p_1 x_1 + p_2 x_2 \\ \text{s.t.} \quad & u(x_1, x_2) \geq \bar{u} \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned} \iff \begin{aligned} \text{Max} \quad & -(p_1 x_1 + p_2 x_2) \\ \text{s.t.} \quad & -u(x_1, x_2) \leq -\bar{u} \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \end{aligned}$$

Hence you can solve the EMP using the same approach developed for UMP.

Of course, the multipliers of the two problems do not have the same interpretations.

The system of FOC relative to each commodity x_ℓ in the two problems at the beginning of the page yields to the same optimal bundle.