The Walrasian Demand Correspondence. The Hicksian Demand Correspondence. The law of demand.

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WARP and the law of demand in UMP

Proposition (WARP)

The Walrasian demand function $x(p, w)$ satisfies WARP if and only if, for any compensated price change from (p, w) to $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

with strict inequality whenever $x(p', w') \neq x(p, w)$.

- The Weak Axiom imposes a consistency requirement on the Walrasian demand, and implies a form of the law of demand, in that the change in prices and in Walrasian demands move in opposite directions for every compensated price change!!

Compensated change in prices and Walrasian demand

- Consider commodity l and the effect on $x_l(p, w)$ of a compensated change in the price of p_l, only.

- $\Delta p = p' - p = (0, \ldots, \Delta p_l, \ldots, 0)$. We want to measure Δx_l, Proposition WARP implies that if $\Delta p_l > 0$ then $\Delta x_l < 0$.

- We cannot say much about the effect of a price change that is not compensated!

- Consider the differential version of Proposition WARP

$$dp \cdot dx \leq 0$$

for a compensated change in wealth induced by a change in the price vector.

Substitution effect and Walrasian demand

- In $dp \cdot dx \leq 0$, dx measures the total variation of the (array of) Walrasian demand $x(p', w' = p' \cdot x(p, w))$ induced by the change in price and the compensation in wealth.

$$dx = D_p x(p, w)dp + D_w x(p, w)dw$$

with $D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \dots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$

and $D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \dots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix}$

Substitution effect and Walrasian demand

- Since we deal with compensated price changes, $dw = dp \cdot x(p, w)$. Hence,

$$dx = D_p x(p, w) dp + D_w x(p, w)[dp \cdot x(p, w)]$$

or

$$dx = [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp.$$

Finally,

$$dp \cdot dx = dp \cdot [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp \leq 0$$

Slutsky matrix and substitution effects

$$[D_p x(p,w) + D_w x(p,w) x(p,w)^T] \equiv S(p,w)$$

is an $(L \text{ x } L)$ matrix, called Slutsky matrix, $S(p,w)$, with generic element of row l and column k equal to

$$s_{lk}(p,w) = [\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)]$$

which is called the substitution effect.

Slutsky matrix and substitution effects

$$s_{lk}(p,w) = [\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)]$$

The substitution effects captures the (differential) change in demand of good l due to a differential change in the price of good k, when wealth is compensated so that the consumer can just afford his original bundle ... hence induced by change in relative prices only.

$\frac{\partial x_l(p,w)}{\partial p_k} dp_k$ measures change in demand of good l if w is unchanged;

$x_k(p,w) dp_k$ measures the compensated change in wealth;

$\frac{\partial x_l(p,w)}{\partial w} [x_k(p,w) dp_k]$ measures the change in demand of good l due to the compensated change in wealth.

Slutsky matrix and substitution effects

- To summarize, $dp \cdot dx \leq 0$ is equivalent to

$$dp \cdot [D_p x(p,w) + D_w x(p,w) x(p,w)^T] dp \leq 0$$

- Since the Walrasian demands satisfy the weak axiom, the Slutsky matrix is negative semi-definite for every (p, w).

- Negative semi-definiteness of $S(p,w)$ implies that $s_{ll}(p,w) \leq 0$ for every $l = 1, 2, ..., L$, own substitution effects are non-positive.

- However, we know that $\frac{\partial x_l(p,w)}{\partial p_l} > 0$ (for Giffen goods), hence for $s_{ll}(p,w) \leq 0$ it has to be $\frac{\partial x_l(p,w)}{\partial w} < 0$. That is, a good can be a Giffen good at some (p,w) only if it is inferior.

The Hicksian demand and the compensated law of demand

Theorem 7

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $h(p, \overline{u})$ uniquely identifies the optimal bundle for all $p \gg 0$. Then, the Hicksian demand function satisfies the compensated law of demand: for all p, p'

$$(p' - p) \cdot \big(h(p', \overline{u}) - h(p, \overline{u})\big) \leq 0$$

The compensated law of demand

Proof of Theorem 7

- By definition, $h(p, \overline{u})$ solves the expenditure minimization problem at price $p \gg 0$. Hence,

$$\begin{aligned} p' \cdot h(p', \overline{u}) &\leq p' \cdot h(p, \overline{u}) \\ p \cdot h(p, \overline{u}) &\leq p \cdot h(p', \overline{u}) \end{aligned}$$

- Rearranging the two inequalities, we get

$$\begin{aligned} p' \cdot \left[h(p', \overline{u}) - h(p, \overline{u})\right] &\leq 0 \\ -p \cdot \left[h(p', \overline{u}) - h(p, \overline{u})\right] &\leq 0 \end{aligned}$$

summing we get the result.■

The compensated law of demand

Proof of Theorem 7

- The result we just proved implies that, differently from the Walrasian demand, the change of the Hicksian demand is *always* inverse with respect to any change in prices.

- The inverse relationship holds for each commodity, i.e.

$$(p' - p) \cdot \big(h(p', \overline{u}) - h(p, \overline{u})\big) \leq 0,$$

  is a compact way to express that

$$(p'_k - p_k) \cdot \big(h_k(p', \overline{u}) - h_k(p, \overline{u})\big) \leq 0 \qquad \text{for } k = 1, 2, \ldots, L.$$

Duality: implications for the value functions

- We have formally shown that EMP is the dual problem of UMP and viceversa.

- More precisely, by Theorem 8, if $u(.)$ is a continuous utility function representing LNS $\succsim$ on $X = \mathbb{R}^L_+$ and if $p \gg 0$,

a) if x^* is optimal in *UMP* at $w > 0$, then x^* is optimal in *EMP* at $u(x^*)$. Moreover, the expenditure function of such *EMP* is exactly equal to w, i.e. $p \cdot x^* = w$;

b) if x^* is optimal in *EMP* at $\overline{u} > u(0)$, then x^* is optimal in *UMP* at wealth equal to $p \cdot x^*$. Moreover, the indirect utility of such *UMP* is exactly equal to $\overline{u}$, i.e. $u(x^*) = \overline{u}$.

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Duality: expenditure function

Theorem 8 supports the following reasoning.

Let $x(p, w)$ be a solution to *UMP* given $p \gg 0$ and $w > 0$, so that

- $p \cdot x(p, w) = w$ (by Walras' law),
- $u(x(p, w)) = v(p, w) \geq \bar{u}$.

Then,

$$e(p, v(p, w)) = p \cdot x(p, w) = w \tag{5}$$

for all $p \gg 0$ and $w > 0$.

An application of duality

Equation (5) states that $e(p, v(p,w)) = w$ for all $p \gg 0$ and $w > 0$.

For example, take $e(p,\bar{u}) = p_1^{\alpha} p_2^{\beta}\ \exp(\bar{u})$. By duality we know that $\bar{u} = v(p,w)$ and that $w = e(p,\bar{u})$. Hence,

$$e(p, v(p,w)) = p_1^{\alpha} p_2^{\beta}\ \exp(v(p,w)) = w$$

Simple computation yields,

$$\exp(v(p,w)) = \frac{w}{p_1^{\alpha} p_2^{\beta}}$$

$$v(p,w) = ln\left(\frac{w}{p_1^{\alpha} p_2^{\beta}}\right) = ln(w) - \alpha ln(p_1) - \beta ln(p_2).$$

Duality: indirect utility

If $h(p, \overline{u})$ is a solution to *EMP* given $p \gg 0$ and $\overline{u} > u(0)$, so that

- $u(h(p, \overline{u})) = \overline{u}$ (no-excess utility),
- $p \cdot h(p, \overline{u}) = e(p, \overline{u}) = w$.

Then,

$$v(p, e(p, \overline{u})) = u(h(p, \overline{u})) = \overline{u} \tag{6}$$

for all $p \gg 0$ and $\overline{u} > u(0)$.

Fix the price vector $p \gg 0$, equations (6) and (5) imply that the indirect utility function and expenditure function are the inverse of one another.

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