

## PRODUCTION SINGLE OUTPUT

- $Y$  exhibits constant returns to scale  
iff (is equivalent to)  $f(\cdot)$  is  
homogeneous of degree one.

$\text{If } Y \text{ has CRS} \Rightarrow f(\cdot) \text{ is homog. of degree one}$

let  $z \in \mathbb{R}_+^{n-1}$  and  $\alpha > 0$ .

$$Y = \{(q, z) \in \mathbb{R}_+^n : q - f(z) \leq 0\}$$

the prod. plan  $(-z, f(z)) \in Y$ .

By constant returns to scale,  
 $(-\alpha z, \alpha f(z)) \in Y$

But  $q - f(z) \leq 0 \Rightarrow \alpha f(z) - f(\alpha z) \leq 0$   
that is

$$(1) \quad \alpha f(z) \leq f(\alpha z) \quad \forall z, \forall \alpha$$

$$\frac{1}{\alpha} f(\alpha z) \leq f\left(\frac{1}{\alpha} \cdot \alpha z\right) = f(z)$$

$$(2) \quad f(\alpha z) \leq \alpha f(z)$$

Hence (1) & (2)  $\Rightarrow f(\alpha z) = \alpha f(z)$

If  $f(\cdot)$  is homog. of degree 1  $\Rightarrow Y$  has CRS

Consider a prod. plan  $(-z, q) \in Y$  and  $\alpha > 0$ .

By def. of  $Y$  :  $q - f(z) \leq 0$  hence

$$q \leq f(z)$$

$$\Rightarrow \alpha q \leq \alpha f(z) = f(\alpha z)$$

Hence,

$$(-\alpha z, \alpha q) \in Y \quad \text{this defines CRS.}$$

## SINGLE OUTPUT TECHNOLOGY

Prop.  $Y$  is convex iff  $f(x)$  is concave.

*If  $Y$  is convex  $\Rightarrow f(x)$  is concave.*

$$Y = \left\{ (-x, q) \in \mathbb{R}^m : q - f(x) \leq 0 \text{ and } x_1, \dots, x_L \geq 0 \right\}$$

take  $\bar{x}, \bar{x}' \in \mathbb{R}_+^{m-1}$  and  $\alpha \in [0, 1]$  s.t.

$$(-\bar{x}, f(\bar{x})) \in Y \text{ and } (-\bar{x}', f(\bar{x}')) \in Y$$

by convexity

$$(-[\alpha \bar{x} + (1-\alpha) \bar{x}'], \alpha \bar{q} + (1-\alpha) \bar{q}') \in Y$$

By def. of prod. set

$$\alpha f(\bar{x}) + (1-\alpha) f(\bar{x}') \leq f(\alpha \bar{x} + (1-\alpha) \bar{x}')$$

that implies  $f(\cdot)$  concave.

*If  $f(\cdot)$  is concave  $\Rightarrow Y$  is convex*

Consider two prod. plans  $(-x, q) \in Y$

and  $(-z', q') \in \mathcal{Y}$  by def.

$$q - f(z) \leq 0 \quad q \leq f(z)$$

$$q' - f(z') \leq 0 \quad q' \leq f(z')$$

Construct the convex combination:

$$\begin{aligned} \alpha q + (1-\alpha)q' &\leq \alpha f(z) + (1-\alpha)f(z') \leq \\ &\leq f(\alpha z + (1-\alpha)z') \end{aligned}$$

hence,

$$\alpha q + (1-\alpha)q' \leq f(\alpha z + (1-\alpha)z')$$

i.e.,

$$(-(\alpha z + (1-\alpha)z'), \alpha q + (1-\alpha)q') \in \mathcal{Y}$$

$\Downarrow$

$$\alpha (-z, q) + (1-\alpha)(-z', q') \in \mathcal{Y}$$

i.e.,  $\mathcal{Y}$  is convex.

When  $\mathcal{Y}$  is convex, PMP has a unique solution and FOC are necessary and sufficient for  $\mathcal{Y}(p)$ .