

Geometric Mean transformation

We start with the definition of the geometric mean of X :

$$M_g(X) = \sqrt[n]{\prod_{i=1}^n X_i} \equiv (X_1 \cdot X_2 \cdot X_3 \cdot \dots \cdot X_n)^{\frac{1}{n}} \quad (1)$$

Taking the natural logarithm (\ln) of both sides:

$$\ln(M_g(X)) = \ln \left((X_1 \cdot X_2 \cdot X_3 \cdot \dots \cdot X_n)^{\frac{1}{n}} \right) \quad (2)$$

Using the property of logarithms that $\ln(a^b) = b \cdot \ln(a)$:

$$\ln(M_g(X)) = \frac{1}{n} \ln(X_1 \cdot X_2 \cdot X_3 \cdot \dots \cdot X_n) \quad (3)$$

Since $\ln(a \cdot b) = \ln(a) + \ln(b)$, we may rewrite:

$$\ln(M_g(X)) = \frac{1}{n} (\ln(X_1) + \ln(X_2) + \ln(X_3) + \dots + \ln(X_n)) \quad (4)$$

Now, let's consider the arithmetic mean of X :

$$M_a(X) = \frac{1}{n} \sum_{i=1}^n X_i \equiv \frac{X_1 + X_2 + X_3 + \dots + X_n}{n}; \quad (5)$$

we rewrite the second term of (4) as:

$$\frac{1}{n} (\ln(X_1) + \ln(X_2) + \ln(X_3) + \dots + \ln(X_n)) = M_a(\ln(X)) \quad (6)$$

and it follows that:

$$\ln(M_g(X)) = M_a(\ln(X)). \quad (7)$$

By exponentiating both sides of equation we get

$$\exp(\ln(M_g(X))) = \exp(M_a(\ln(X))) \quad (8)$$

Using the property of exponentials that $\exp(\ln(a)) = a$, we find that the geometric mean of X is equal to the exponential of the arithmetic mean of the logarithm of X :

$$M_g(X) = \exp(M_a(\ln(X))). \quad (9)$$