Introduction to Probability: experiment, events, operations with events and Venn diagrams.

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Outline

- Definitions and instruments:
  - experiment
  - events
  - operations with events
  - Venn diagrams
- Probability measures:
  - Joint, marginal, and conditional probabilities
  - Independency
  - Bayes Theorem

Basic Elements of Probability Theory: Key Concepts

Probability theory deals with **random experiments**, or random trials. A random experiment is such that:

- the possible outcomes are known prior to its conduct
- the outcome is know at the end of the experiment and cannot be predicted with certainty in advance
- it can be repeated any number of times under the same conditions

The last item is relevant only for one approach to assigning probabilities (the frequentist approach), as we shall see.

Basic Elements of Probability Theory: Key Concepts

- The **Sample Space** (Ω): set of possible outcomes (ω) of the experiment.
- The **$\boldsymbol{\sigma}$-algebra** ($\mathcal{F}$) is a collection of subsets ($A \subset \Omega$) that satisfies certain properties: an event A occurs if the outcome ω is an element of A.
- The **Probability Measure** (P): is a function with range [0, 1] defined on $\mathcal{F}$ that satisfies certain properties.

Probability Space

The triple (Ω, $\mathcal{F}$, P) is called a **probability space**.

Experiments and Outcomes

Experiment

A process that, when performed, results in one of many uncertain outcomes.

- Examples:
  - Toss of a coin
  - Roll of a die
  - Selecting a worker

Outcomes

The results of an experiment.

- Examples:
  - Toss of a coin: Head, Tail
  - Roll of a die: 1, 2, 3, 4, 5, 6
  - Selecting a worker: Male, Female

Sample Space

Definition

The *sample space*, denoted as Ω, is the set of all possible outcomes of a random experiment.

Example

- Tossing a fair six-sided die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Flipping a fair coin: $\Omega = \{H, T\}$.

Experiment and Sample Space

Experiment	Sample space
Toss of a coin	$\{H, T\}$
Two successive tosses of a coin	$\{(HH), (HT), (TT), (TH)\}$
Rolling a die	$\{1,2,3,4,5,6\}$
Rolling two dice	$\{(1,1),(1,2),\ldots,(1,6),$ $(2,1),\ldots,(2,6),\ldots,$ $(6,1),\ldots,(6,2),\ldots,(6,6)\}$
University exam	$\{$Fail $(<18), 18, 19, \ldots, 30, 30$ c.l.$\}$
Number of clients in a day	$\{0,1,2,\ldots,\}$
Duration of a bulb	$R^+ = [0,\infty)$
Daily return of a stock	$R = (-\infty,\infty)$

The Empty Set ($\emptyset$)

Definition

The *empty set*, denoted as $\emptyset$ or $\{\}$, is a set that contains no elements, often referred to as the null set.

- The empty set is unique because it has no elements, and its size is zero.
- In the context of probability, it represents an event with zero probability of occurring.

Example

In the context of rolling a fair six-sided die, an example of an event that corresponds to the empty set is the event "rolling a 7."

Event

An event is a subset of Ω. We distinguish between:

- Sample Points or Elementary Events;
- Compound or Composite Events.

Sample Point or Elementary Event

Definition

A *sample point* or *elementary event*, denoted as ω, is an individual outcome within the sample space. It represents a specific result of the random experiment.

- The sample space, Ω, encompasses all potential outcomes.
- A sample point, ω, is a single, particular outcome within that space.

Example

Sample Point (ω) Let's consider the rolling of a fair six-sided die. Each face of the die is a sample point within the sample space (Ω):

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

A sample point, ω, represents a single outcome, e.g., $\omega = 3$ represents rolling a 3.

Compound or Composite Events

Definition

A compound or composite event is obtained by combining elementary events.

$$A = \{\omega \in \Omega : \omega \in A\}.$$

- An event A is a subset of a sample space Ω, denoted as $A \subset \Omega$.
- It is formed by combining its elementary events.
- We say that the event A occurs if the outcome of the experiment, $w \in \Omega$, is an element of the set A.

Example

- Consider the rolling of a six-sided die.
- Sample space: $\Omega = \{\omega_i = i, i = 1, \ldots, 6\}$.
- The event "an even number on the upper side of a die" is the composite event: $A = \{2, 4, 6\}$.

Complementary Event

Definition

The *complementary event*, denoted as A' or $\bar{A}$, is the event that consists of all outcomes in the sample space Ω that are not in event A:

$$\bar{A} = \{\omega \in \Omega : \omega \notin A\}.$$

Example

Consider the sample space for a coin toss: $\Omega = \{H, T\}$.

- If A is the event "getting heads," then the complementary event is $\bar{A} = \{T\}$.
- Similarly, $\bar{A}$ represents the event "not getting heads."

- The complementary event represents the opposite of the given event.
- It is often used to calculate the probability of an event and its complement.

Union of Events

Definition

The *union* of two events, denoted as $A \cup B$, is the event that consists of outcomes that are in either event A or event B, or both:

$$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}.$$

Example

Consider the sample space for flipping a fair coin:

- Event A represents getting heads: $A = \{H\}$.
- Event B represents getting tails: $B = \{T\}$.
- The union of events A and B is $A \cup B = \{H, T\}$, representing outcomes of either getting heads or getting tails.

- The union of events combines outcomes from both events.

Intersection of Events

Definition

The *intersection* of two events, denoted as $A \cap B$, is the event that consists of outcomes that are in both events A and B:

$$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}.$$

Example

Consider the sample space for rolling a fair six-sided die:

- Event A represents rolling an even number: $A = \{2, 4, 6\}$.
- Event B represents rolling a number greater than 3: $B = \{4, 5, 6\}$.
- The intersection of events A and B is $A \cap B = \{4, 6\}$, representing outcomes that satisfy both conditions.

- The intersection of events corresponds to outcomes that meet criteria in both events.

Union of Non-Disjoint Events

Definition

When events A and B share common outcomes, they are non-disjoint or overlapping. In other words, $A \cap B \neq \emptyset$.

Example

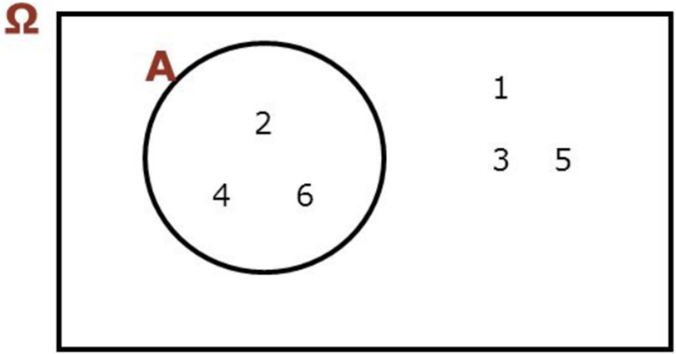
Consider the sample space for drawing a card from a standard deck:

- Event A represents rolling an even number: $A = \{2, 4, 6\}$.
- Event B represents rolling a number greater than 3: $B = \{4, 5, 6\}$.
- The union of events A and B is

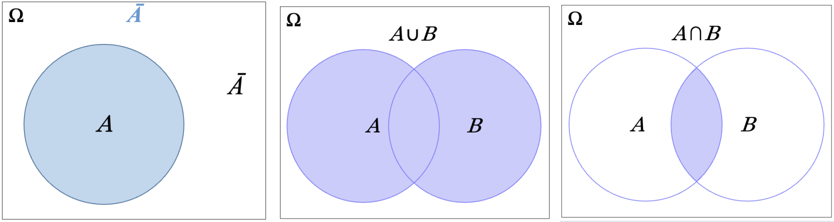
$$A \cup B = A + B - A \cap B.$$

- The union of non-disjoint events combines outcomes, considering common elements once.

Venn Diagram



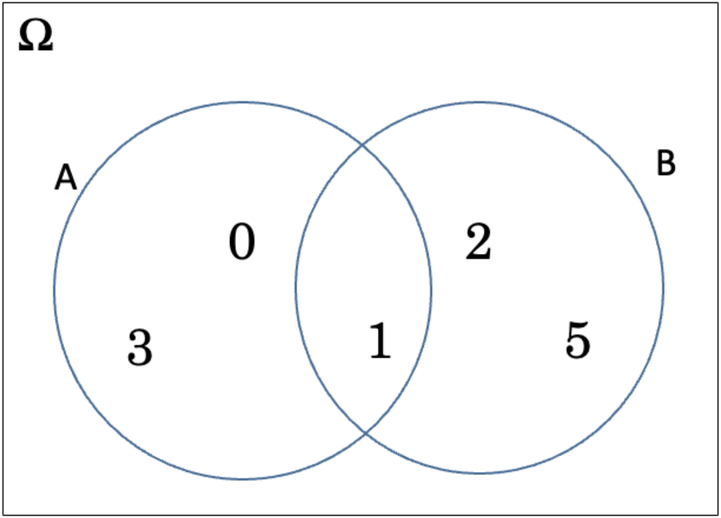
Algebra of Events



Note that:

- $A \cup B = A + B - A \cap B$

Example



Decomposition of an Event

Decomposition Principle

Events can be decomposed by conditioning on two complementary events. For any event A, you can express it as:

$$A = (A \cap B) \cup (A \cap \bar{B})$$

where:

- A is the original event.
- B and $\bar{B}$ are complementary events.
- $(A \cap B)$ is the part of A that occurs when B happens.
- $(A \cap \bar{B})$ is the part of A that occurs when $\bar{B}$ happens.

Decomposition of an Event

Example

Suppose we are selecting cards from a deck. Let:

- A be the event of drawing a red card.
- B be the event of drawing a heart.
- $\bar{B}$ be the event of drawing a non-heart.

We can express A as:

$$A = (A \cap B) \cup (A \cap \bar{B})$$

This decomposition allows us to consider the different ways A can occur based on the occurrence of B or $\bar{B}$.

Summary and some insights

- A composite event can be written as the union of elementary events.
- Unions and intersections are related by the de Morgan's laws:

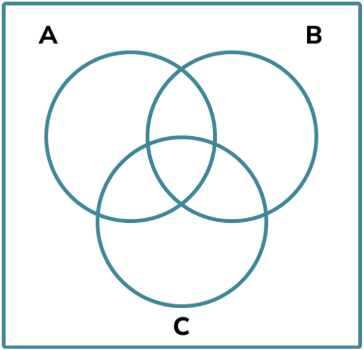
$$A \cup B = \overline{\bar{A} \cap \bar{B}}$$

$$A \cap B = \overline{\bar{A} \cup \bar{B}}$$

- The complement of Ω is the impossible event, denoted $\emptyset$ (empty space).
- The certain event is the event which always occurs. It is complementary to $\emptyset$ and contains all the possible outcomes of the experiment. Moreover, $A \cup \overline{A} = \Omega$.

Properties of the operations between events

Property	**Union**	**Intersection**
Idempotency	$A \cup A = A$	$A \cap A = A$
Neutral event	$A \cup \emptyset = A$	$A \cap \Omega = A$
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$



Mutually Exclusive or Disjoint Events

Definition

Events A and B are said to be mutually exclusive if their intersection is an empty set, i.e., if they cannot occur simultaneously. Formally, we say:

$$A \cap B = \emptyset$$

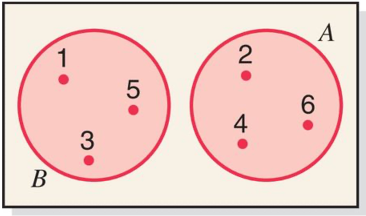
- Disjoint events, also known as mutually exclusive events, are events that cannot occur simultaneously. In other words, if one of the events happens, the other(s) cannot.
- For example, when rolling a six-sided die, the events "getting an even number" and "getting an odd number" are disjoint, as the two sets of outcomes do not overlap.
- Elementary events $w \in \Omega$ are disjoint. For instance, if you toss a coin, either Head or Tail can occur, but not both.

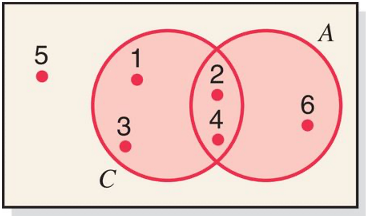
Mutually Exclusive or Disjoint Events

Example

Ex: Roll a die once and define the following events:

- A: An even number is observed $= \{2, 4, 6\}$
- B: An odd number is observed $= \{1, 3, 5\}$
- C: A number less than 5 is observed $= \{1, 2, 3, 4\}$





Partition of Events

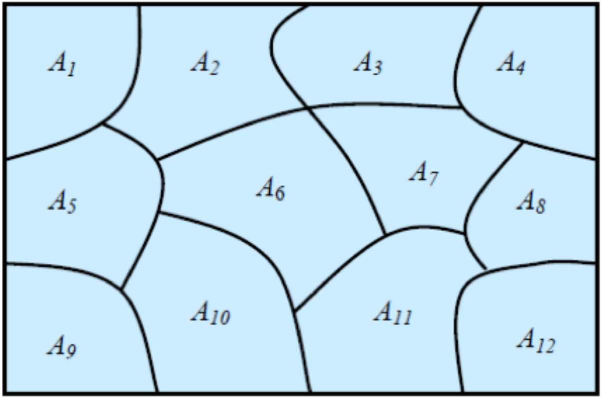
Definition

A partition of the sample space Ω is a collection of events $A_1, A_2, \ldots, A_n$ such that:

- The events are mutually exclusive (disjoint), meaning $A_i \cap A_j = \emptyset$ for $i \neq j$.
- The events are collectively exhaustive, meaning $\bigcup_{i=1}^{n} A_i = \Omega$.

- A partition of events divides the sample space into non-overlapping and exhaustive components.
- It is a useful concept in probability and statistics for simplifying calculations and analysis.

Partition of Events



Sigma Algebra

Definition

A *sigma algebra*, denoted as $\mathcal{F}$, is a collection of subsets of a sample space Ω that satisfies the following properties:

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then the complement $\bar{A}$ is also in $\mathcal{F}$.
3. If $A_1, A_2, A_3, \ldots$ is a countable sequence of sets in $\mathcal{F}$, their union $\cup_{i=1}^{\infty} A_i$ is in $\mathcal{F}$.

- Sigma algebras are a fundamental concept in probability theory, providing a structure for defining measurable events.
- They ensure that we can work with well-defined probabilities for various events.

Sigma Algebra: Example

Example

For the experiment of rolling a fair six-sided die. The space of events is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and we define a sigma algebra $\mathcal{F}$ as:

$$\mathcal{F} = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}$$

- $\mathcal{F}$ contains the sample space,its complements, and other combinations of die roll outcomes.
- It satisfies the properties of a sigma algebra:
  - $\Omega \in \mathcal{F}$
  - $\bar{\Omega} = \emptyset \in \mathcal{F}$
  - Let define $A = \{1, 3, 5\}$. Then, $\bar{A} = \{2, 4, 6\}$.
  - $\{1, 3, 5\} \cup \{2, 4, 6\} = \Omega \in \mathcal{F}$

Counting Rule

When an experiment has a large number of outcomes, it may not be practical to list them all. In such cases, the counting rule provides a method for determining the total number of outcomes.
If an experiment consists of k steps, each with n_i outcomes, then the total number of final outcomes is given by the product of the possible outcomes at each step:

$$\text{Total Outcomes} = n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

For example, consider the experiment of tossing a coin (2 outcomes) 6 times. The total number of outcomes is calculated as follows:

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$$

Factorials

Definition

The factorial of a number is obtained by multiplying all the integers from that number down to 1. It is denoted as $n!$ and calculated as:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

Example

For instance, the factorial of 5 is calculated as follows:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Note:
A conventional rule in mathematics is that $0! = 1$.

Permutations

Definition

Permutation is the arrangement of elements where the order of elements matters. It involves arranging digits, numbers, alphabets, colors, or letters, taking some or all of them at a time.
Permutations are typically represented as:

$$P_{n,x} = \frac{n!}{(n-x)!}$$

Example

Three members need to be randomly chosen from a set of five. To calculate the number of permutations possible, we can use the permutation formula. In this case, $n = 5$ and $x = 3$. So, the number of permutations is:

$$P_{5,3} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 60$$

Permutations with Repetition

Definition

In a permutation with repetition, it's possible to select the same unit multiple times. The formula for calculating permutations with repetition is:

$$P_{n,x} = n^x$$

Example

For instance, when 3 members need to be randomly chosen from a set of 5 with repetition, the number of permutations is calculated as:

$$P_{5,3} = 5^3 = 5 \cdot 5 \cdot 5 = 125$$

Combinations

Definition

Combination is the process of selecting elements where the order of selection does not matter. This involves choosing from digits, numbers, alphabets, colors, or letters, taking some or all of them at a time.
Combinations are typically represented as:

$$C_{n,x} = \frac{n!}{x!(n-x)!} \quad \text{or} \quad C_{n,x} = \binom{n}{x}$$

Note that:

- $n \geq x$
- $C_{n,n} = 1$
- $C_{n,0} = 1$

Combinations: Example

Example

Members of a faculty committee need to be randomly chosen from a set of 5 candidates. How many combinations are possible?
The number of combinations, denoted as $C_{n,x}$, is given by:

$$C_{n,x} = \frac{n!}{x!(n-x)!}$$

For selecting 3 members from 5 candidates, we have:

$$C_{5,3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

The possible combinations are: ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, CDE, BDE. Remember, in combinations, the order is not important. Thus, ABC is equivalent to ACB and BCA.

Combinations vs. Permutations

Combinations

- Selection of objects where order does not matter.
- Denoted as $C_{n,x}$.
- Formula: $C_{n,x} = \frac{n!}{x!(n-x)!}$.

Permutations

- Arrangement of objects in a definite order.
- Denoted as $P_{n,x}$.
- Formula: $P_{n,x} = \frac{n!}{(n-x)!}$.

Example

- Combining elements in a lottery draw (combinations).
- Arranging students in a row for a photo (permutations).