Properties of Estimators

Rosario Barone

Tor Vergata University of Rome

Statistical tools for decision making

Undergraduate Degree in Global Governance

A.Y. 2024/2025

Properties of Estimators

- Unbiasedness
- Efficiency
- Consistency

Unbiasedness

Unbiased Estimator

An estimator $\hat{\theta}$ is unbiased if its expected value is equal to the true value of the parameter being estimated. That is, $\mathrm{E}(\hat{\theta}) = \theta$.

- Let's consider an estimator $\hat{\theta}$ for the parameter θ.
- The bias of the estimator is defined as the difference between its expected value and the true parameter value: $\mathrm{Bias}(\hat{\theta}) = \mathrm{E}(\hat{\theta}) - \theta$.
- $\mathrm{E}(\hat{\theta}) = \int \hat{\theta} f(\mathbf{x};\theta) d\mathbf{x}$, where $f(\mathbf{x};\theta)$ is the probability density function (pdf) of the observed data $\mathbf{x}$.
- If the bias is zero, i.e., $\mathrm{Bias}(\hat{\theta}) = 0$, the estimator is unbiased.

Consistency

Consistent Estimator

An estimator $\hat{\theta}$ is consistent if it converges in probability to the true value of the parameter as the sample size increases. That is, $\hat{\theta} \xrightarrow{\mathrm{P}} \theta$.

- Let $\hat{\theta}$ be an estimator for the parameter θ.
- Convergence in probability means that as the sample size increases, the probability that $\hat{\theta}$ deviates from θ becomes smaller.
- Formally, for any $\epsilon > 0$, $\lim_{n\to\infty} P(|\hat{\theta} - \theta| > \epsilon) = 0$.
- A consistent estimator provides increasingly accurate estimates as more data becomes available.

Efficiency

Efficient Estimator

Among unbiased estimators, an estimator $\hat{\theta}_1$ is more efficient than another estimator $\hat{\theta}_2$ if it has a smaller variance for all sample sizes. That is, $\mathrm{Var}(\hat{\theta}_1) < \mathrm{Var}(\hat{\theta}_2)$.

- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators for the parameter θ.
- The efficiency of an estimator is determined by its variance.
- An efficient estimator achieves the smallest possible variance among all unbiased estimators, making it more precise.
- The **Cramér-Rao lower bound** provides a lower bound for the variance of any unbiased estimator.

Cramer-Rao Inequality

- The Cramer-Rao Inequality relates the variance of an unbiased estimator to the Fisher Information.
- Let $\hat{\theta}$ be an unbiased estimator for a parameter θ.
- The Cramer-Rao Inequality states:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{\mathcal{I}(\theta)},$$

  where $\mathcal{I}(\theta)$ is the Fisher Information.
- The Fisher Information (will be introduced later) measures the amount of information that the data provides about the parameter.

Mean Squared Error

- The Mean Squared Error (MSE) is a commonly used measure to assess the quality of an estimator.
- It quantifies the average squared difference between the estimated values and the true values of a parameter.
- The Mean Squared Error is defined as the expected value of the squared difference between the estimator $\hat{\theta}$ and the true parameter value θ:
$$MSE(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right]$$

Properties of Mean Squared Error

The Mean Squared Error possesses several important properties:

- **Non-Negativity**: The MSE is always non-negative: $MSE(\hat{\theta}) \geq 0$.
- **Bias-Variance Decomposition**: The MSE can be decomposed into the sum of the squared bias and the variance of the estimator:

  $$MSE(\hat{\theta}) = \text{Bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta})$$

  where $\text{Bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta$ is the bias and $\text{Var}(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2\right]$ is the variance of the estimator.
- **Efficiency**: An efficient estimator minimizes the MSE among all unbiased estimators.

Bias-Variance Decomposition

Theorem

The MSE can be decomposed into the sum of the squared bias and the variance of the estimator:

$$MSE(\hat{\theta}) = Bias(\hat{\theta})^2 + Var(\hat{\theta})$$

Proof:

$$\begin{aligned}
MSE(\hat{\theta}) &= \mathbb{E}\left[(\hat{\theta} - \theta)^2\right] \\
&= \mathbb{E}\left[(\hat{\theta} - \mathbb{E}(\hat{\theta}) + \mathbb{E}(\hat{\theta}) - \theta)^2\right] \\
&= \underbrace{\mathbb{E}\left[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2\right]}_{\text{Var}(\hat{\theta})} + 2 \cdot \mathbb{E}\left[\hat{\theta} - \mathbb{E}(\hat{\theta})\right] \cdot \mathbb{E}\left[\mathbb{E}(\hat{\theta}) - \theta\right] + \underbrace{\mathbb{E}\left[(\mathbb{E}(\hat{\theta}) - \theta)^2\right]}_{\text{Bias}(\hat{\theta})^2} \\
&= \text{Var}(\hat{\theta}) + \text{Bias}(\hat{\theta})^2
\end{aligned}$$

Note: $\mathbb{E}\left[\hat{\theta} - \mathbb{E}(\hat{\theta})\right]$

Standard Errors

- The standard error (SE) is a measure of the variability of an estimator.
- It represents the standard deviation of the sampling distribution of the estimator.
- They are typically used to construct confidence intervals and perform hypothesis tests.
- A smaller standard error indicates a more precise estimator.
- Estimation of Standard Errors:
  - Analytical Methods
  - Numerical Methods

Unbiasedness of Sample Mean Estimator

Unbiased Estimator

An estimator $\hat{\theta}$ is unbiased if its expected value is equal to the true value of the parameter being estimated. That is, $\mathrm{E}(\hat{\theta}) = \theta$.

Using linearity of expectation and properties of random variables, we obtain

$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu.$

Variance of $\bar{X}$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) =$$

$$= \frac{1}{n^2} \cdot \mathrm{Var}(X_1 + X_2 + \ldots + X_n) =$$

$$= \frac{1}{n^2} \cdot (\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)) \quad \text{(by independence)}$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 \quad (\text{since } \mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \ldots = \mathrm{Var}(X_n) = \sigma^2) =$$

$$= \frac{\sigma^2}{n}$$

Consistency and Efficiency of Sample Mean Estimator

Consistent Estimator

The estimator $\bar{X}$ is consistent if it converges in probability to μ as the sample size increases. That is, $\bar{X} \xrightarrow{\mathrm{P}} \mu$, i.e. $\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$ for any $\epsilon > 0$.

Efficient Estimator

Among unbiased estimators, the sample mean estimator $\bar{X}$ is more efficient than another estimator $\hat{\theta}$. That is, $\mathrm{Var}(\bar{X}) < \mathrm{Var}(\hat{\theta})$.

Sample Variance Estimator

Let's consider a random sample of size n from a population with unknown variance σ^2. The biased variance estimator, denoted as S_b^2, is calculated using the following formula:

$$S_b^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

where X_i represents individual observations, and $\bar{X}$ is the sample mean. It can be proved that:

$$\mathbb{E}[S_b^2] = \frac{n-1}{n}\sigma^2$$

Bias correction: unbiased Sample Variance Estimator

- S_b^2 underestimates the true population variance.
- The bias decreases as the sample size increases.

Let's consider a random sample of size n from a population with unknown variance σ^2. The sample variance estimator, denoted as S^2, is calculated using the following formula:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

where X_i represents individual observations, and $\bar{X}$ is the sample mean. It can be proved that

$$\mathbb{E}[S^2] = \sigma^2.$$