

Properties of Estimators

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Properties of Estimators

- Unbiasedness
- Efficiency
- Consistency

Unbiasedness

Unbiased Estimator

An estimator $\hat{\theta}$ is unbiased if its expected value is equal to the true value of the parameter being estimated. That is, $E(\hat{\theta}) = \theta$.

- Let's consider an estimator $\hat{\theta}$ for the parameter θ .
- The bias of the estimator is defined as the difference between its expected value and the true parameter value: $\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$.
- $E(\hat{\theta}) = \int \hat{\theta} f(\mathbf{x}; \theta) d\mathbf{x}$, where $f(\mathbf{x}; \theta)$ is the probability density function (pdf) of the observed data \mathbf{x} .
- If the bias is zero, i.e., $\text{Bias}(\hat{\theta}) = 0$, the estimator is unbiased.

Consistency

Consistent Estimator

An estimator $\hat{\theta}$ is consistent if it converges in probability to the true value of the parameter as the sample size increases. That is, $\hat{\theta} \xrightarrow{P} \theta$.

- Let $\hat{\theta}$ be an estimator for the parameter θ .
- Convergence in probability means that as the sample size increases, the probability that $\hat{\theta}$ deviates from θ becomes smaller.
- Formally, for any $\epsilon > 0$, $\lim_{n \rightarrow \infty} P(|\hat{\theta} - \theta| > \epsilon) = 0$.
- A consistent estimator provides increasingly accurate estimates as more data becomes available.

Efficiency

Efficient Estimator

Among unbiased estimators, an estimator $\hat{\theta}_1$ is more efficient than another estimator $\hat{\theta}_2$ if it has a smaller variance for all sample sizes. That is, $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$.

- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators for the parameter θ .
- The efficiency of an estimator is determined by its variance.
- An efficient estimator achieves the smallest possible variance among all unbiased estimators, making it more precise.
- The **Cramér-Rao lower bound** provides a lower bound for the variance of any unbiased estimator.

Cramer-Rao Inequality

- The Cramer-Rao Inequality relates the variance of an unbiased estimator to the Fisher Information.
- Let $\hat{\theta}$ be an unbiased estimator for a parameter θ .
- The Cramer-Rao Inequality states:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{\mathcal{I}(\theta)},$$

where $\mathcal{I}(\theta)$ is the Fisher Information.

- The Fisher Information (will be introduced later) measures the amount of information that the data provides about the parameter.

Mean Squared Error

- The Mean Squared Error (MSE) is a commonly used measure to assess the quality of an estimator.
- It quantifies the average squared difference between the estimated values and the true values of a parameter.
- The Mean Squared Error is defined as the expected value of the squared difference between the estimator $\hat{\theta}$ and the true parameter value θ :

$$MSE(\hat{\theta}) = \mathbb{E} \left[(\hat{\theta} - \theta)^2 \right]$$

Properties of Mean Squared Error

The Mean Squared Error possesses several important properties:

- **Non-Negativity:** The MSE is always non-negative: $MSE(\hat{\theta}) \geq 0$.
- **Bias-Variance Decomposition:** The MSE can be decomposed into the sum of the squared bias and the variance of the estimator:

$$MSE(\hat{\theta}) = \text{Bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta})$$

where $\text{Bias}(\hat{\theta}) = \mathbb{E}(\hat{\theta}) - \theta$ is the bias and $\text{Var}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2]$ is the variance of the estimator.

- **Efficiency:** An efficient estimator minimizes the MSE among all unbiased estimators.

Bias-Variance Decomposition

Theorem

The MSE can be decomposed into the sum of the squared bias and the variance of the estimator:

$$MSE(\hat{\theta}) = Bias(\hat{\theta})^2 + Var(\hat{\theta})$$

Proof:

$$\begin{aligned} MSE(\hat{\theta}) &= \mathbb{E} \left[(\hat{\theta} - \theta)^2 \right] \\ &= \mathbb{E} \left[(\hat{\theta} - \mathbb{E}(\hat{\theta}) + \mathbb{E}(\hat{\theta}) - \theta)^2 \right] \\ &= \underbrace{\mathbb{E} \left[(\hat{\theta} - \mathbb{E}(\hat{\theta}))^2 \right]}_{Var(\hat{\theta})} + 2 \cdot \mathbb{E} \left[\hat{\theta} - \mathbb{E}(\hat{\theta}) \right] \cdot \mathbb{E} \left[\mathbb{E}(\hat{\theta}) - \theta \right] + \underbrace{\mathbb{E} \left[(\mathbb{E}(\hat{\theta}) - \theta)^2 \right]}_{Bias(\hat{\theta})^2} \\ &= Var(\hat{\theta}) + Bias(\hat{\theta})^2 \end{aligned}$$

Note: $\mathbb{E} \left[\hat{\theta} - \mathbb{E}(\hat{\theta}) \right]$

Standard Errors

- The standard error (SE) is a measure of the variability of an estimator.
- It represents the standard deviation of the sampling distribution of the estimator.
- They are typically used to construct confidence intervals and perform hypothesis tests.
- A smaller standard error indicates a more precise estimator.
- Estimation of Standard Errors:
 - ▶ Analytical Methods
 - ▶ Numerical Methods

Unbiasedness of Sample Mean Estimator

Unbiased Estimator

An estimator $\hat{\theta}$ is unbiased if its expected value is equal to the true value of the parameter being estimated. That is, $E(\hat{\theta}) = \theta$.

Using linearity of expectation and properties of random variables, we obtain

$$E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1}{n} \sum_{i=1}^n \mu = \mu.$$

Variance of \bar{X}

$$\begin{aligned}\text{Var}(\bar{X}) &= \text{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \\&= \frac{1}{n^2} \cdot \text{Var}(X_1 + X_2 + \dots + X_n) = \\&= \frac{1}{n^2} \cdot (\text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)) \quad (\text{by independence}) \\&= \frac{1}{n^2} \cdot n \cdot \sigma^2 \quad (\text{since } \text{Var}(X_1) = \text{Var}(X_2) = \dots = \text{Var}(X_n) = \sigma^2) = \\&= \frac{\sigma^2}{n}\end{aligned}$$

Consistency and Efficiency of Sample Mean Estimator

Consistent Estimator

The estimator \bar{X} is consistent if it converges in probability to μ as the sample size increases. That is, $\bar{X} \xrightarrow{P} \mu$, i.e. $\lim_{n \rightarrow \infty} P(|\bar{X}_n - \mu| > \epsilon) = 0$ for any $\epsilon > 0$.

Efficient Estimator

Among unbiased estimators, the sample mean estimator \bar{X} is more efficient than another estimator $\hat{\theta}$. That is, $\text{Var}(\bar{X}) < \text{Var}(\hat{\theta})$.

Sample Variance Estimator

Let's consider a random sample of size n from a population with unknown variance σ^2 . The biased variance estimator, denoted as S_b^2 , is calculated using the following formula:

$$S_b^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$$

where X_i represents individual observations, and \bar{X} is the sample mean. It can be proved that:

$$\mathbb{E}[S_b^2] = \frac{n-1}{n} \sigma^2$$

Bias correction: unbiased Sample Variance Estimator

- S_b^2 underestimates the true population variance.
- The bias decreases as the sample size increases.

Let's consider a random sample of size n from a population with unknown variance σ^2 . The sample variance estimator, denoted as S^2 , is calculated using the following formula:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

where X_i represents individual observations, and \bar{X} is the sample mean. It can be proved that

$$\mathbb{E}[S^2] = \sigma^2.$$