Mock Exam
(version suitable for the students who attended classworks)
Time: 1 hour and 30 minutes

Katia Colaneri
Asset Pricing - Master program in Finance and Banking

October 29, 2024

Question 1

State and prove the result that guarantees that the binomial one period market model is arbitrage free.

The answer to this question is left to the students

Exercise 1

Consider a binomial 2-period model with the following data: $S_0 = 100$, $B_0 = 1$, $u = 1.1$, $d = 0.9$, $r = 0.05$.

1. Compute the price and the hedging strategy of a European Put option with strike $K = 105$ and maturity $T = 2$.
2. Compute the price of an American Put option with strike $K = 105$ and maturity $T = 2$.
3. How do you justify the difference between the prices?

Short solutions:

The market is arbitrage free and complete. The risk neutral measure is characterised by $q = \frac{3}{4}$ and $1 - q = \frac{1}{4}$.

1. We use backward induction. At $t = 1$,
   - if $S_1 = 110$, $\Pi^E(1, X) = \frac{1}{1.05}\left(0 \cdot \frac{3}{4} + 6 \cdot \frac{1}{4}\right) = 1.428$ and the hedging strategy is $(h_2^0, h_2^1) = (33/(1.05)^2, -6/22)$

- if $S_1 = 90$, $\Pi^E(1, X) = \frac{1}{1.05}\left(6 \cdot \frac{3}{4} + 24 \cdot \frac{1}{4}\right) = 10$ and the hedging strategy is $(h_2^0, h_2^1) = (105/(1.05)^2, -1)$

   At $t = 0$, $\Pi(0, X) = \frac{1}{1.05}\left(1.428 \cdot \frac{3}{4} + 10 \cdot \frac{1}{4}\right) \sim 3.4$ and the hedging strategy is $(h_2^0, h_2^1) = (42.26, -0.4286)$.

2. We use backward induction. At $t = 1$,

   - if $S_1 = 110$, $\Pi(1, X) = \max\left(0, \frac{1}{1.05}\left(0 \cdot \frac{3}{4} + 6 \cdot \frac{1}{4}\right)\right) = 1.428$
   - if $S_1 = 90$, $\Pi(1, X) = \max\left(15, \frac{1}{1.05}\left(6 \cdot \frac{3}{4} + 24 \cdot \frac{1}{4}\right)\right) = 15$

   At $t = 0$, $\Pi(0, X) = \max\left(5, \frac{1}{1.05}\left(1.428 \cdot \frac{3}{4} + 15 \cdot \frac{1}{4}\right)\right) = 5$

3. The answer to this question is left to the students

Exercise 2

In the Black and Scholes market model consider an option with underlying $(S_t)_{t \geq 0}$, with the values $S_0 = 20$, $\mu = 0.16$, $\sigma = 0.36$ and $r = 0.04$. Compute the price of an option with payoff $\phi(S_T) = (\ln(S_T^2) - K)^+$, with $K = 6$ and $T = 1$.

Short solutions:
Recall that under the real world probability P, the stock price satisfies

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \quad S_0 = s$$

and under the risk neutral probability Q it satisfies

$$dS_t = S_t r dt + S_t \sigma d\tilde{W}_t, \quad S_0 = s$$

In particular, under Q it holds that

$$S_T = S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)}$$

and

$$\ln(S_T) = \ln(s) + (r - \frac{1}{2}\sigma^2)T + \sigma \tilde{W}_T.$$

Note that in particular, pugging the values of the parameters we get that

$$\ln(S_T) = 2.97 + 0.36Z$$

where $Z \sim N(0, 1)$.
Now we are ready to compute the the price of the option using Feynman-Kac formula:

$$
\begin{aligned}
\Pi(0,X) &= e^{-rT}\mathbb{E}^Q\left[\Phi(S_T)|S_0=20\right] = e^{-r(T-t)}\mathbb{E}^Q\left[(2\ln S_T-K)^+|S_0=20\right] \\
&= e^{-rT}\mathbb{E}^Q\left[(2\ln S_T-K)\mathbf{1}_{\ln S_T>K/2}|S_t=20\right] \\
&= e^{-0.04}\ \mathbb{E}^Q\left[(5.94+0.72Z-6)\mathbf{1}_{Z\geq 0.083}\right] \\
&= e^{-0.04}\ 0.72\int_{0.083}^{+\infty} z\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz - e^{-0.04}\ 0.06\ Q(Z\geq 0.083) \\
&= e^{-0.04}\ 0.72\ e^{-(0.083)^2/2} - e^{-0.04}\ 0.06\ N(-0.083)
\end{aligned}
$$

where $N(\cdot)$ is the cumulative distribution of the standard normal.