OPTIMISATION from GRAVELL-REES

Ex.
$$\max_{x_1, x_2, \dots x_L} u(x_1, x_2, \dots, x_L)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 + \dots + p_L x_L \leq w \quad (1)$$

$$x_\ell \geq 0 \quad \forall \ell = 1, \dots, L \quad (2)$$

$u(\cdot)$ is called objective function

$x_1, x_2, \dots, x_L$ choice variables

feasible set $\Rightarrow$ available alternatives.

A solution $x^* = (x_1^*, \dots, x_L^*) \in \mathbb{R}^L_+$ is a vector/profile of commodities with the properties that:

$$x^* \in B(p, w) \text{ and}$$

$$u(x^*) \geq u(y) \qquad \forall\, y \in B(p,w)$$
$$\& \text{ s.t. } y_\ell \geq 0 \quad \forall \ell$$

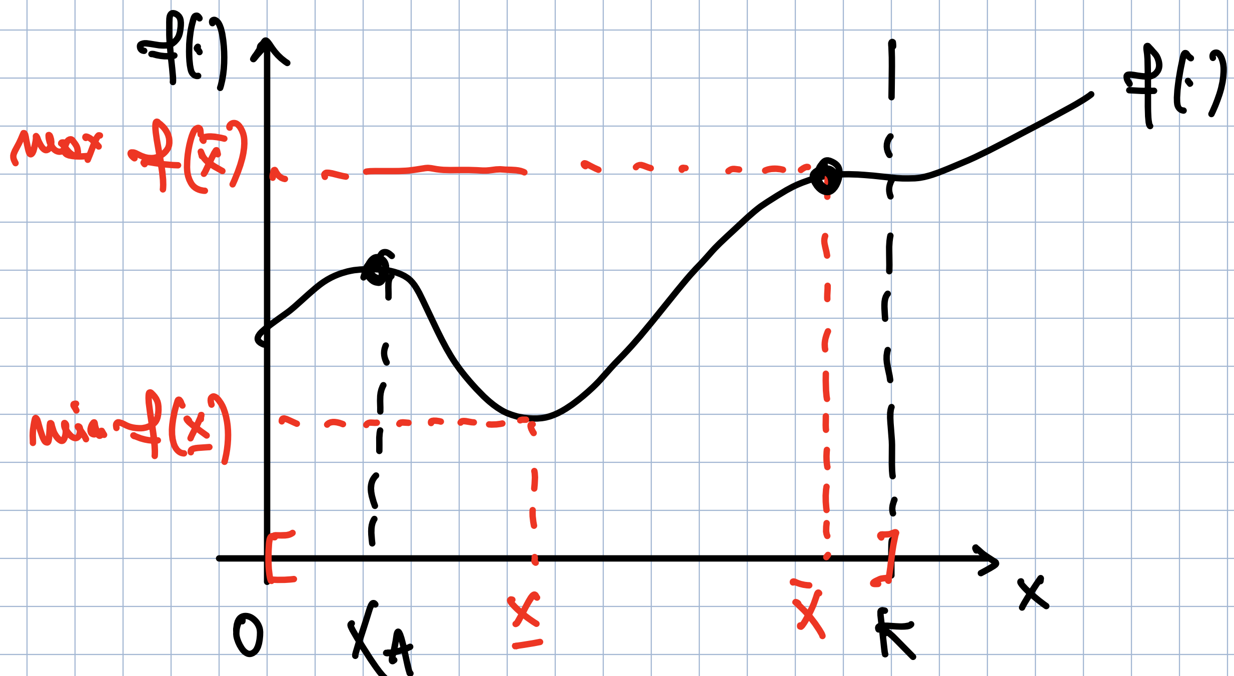
$$x^* \in \arg\max_{\tilde{x}} u(\tilde{x}) \quad \text{s.t. } (1)-(2)$$

Our problem is to find such an x^*.

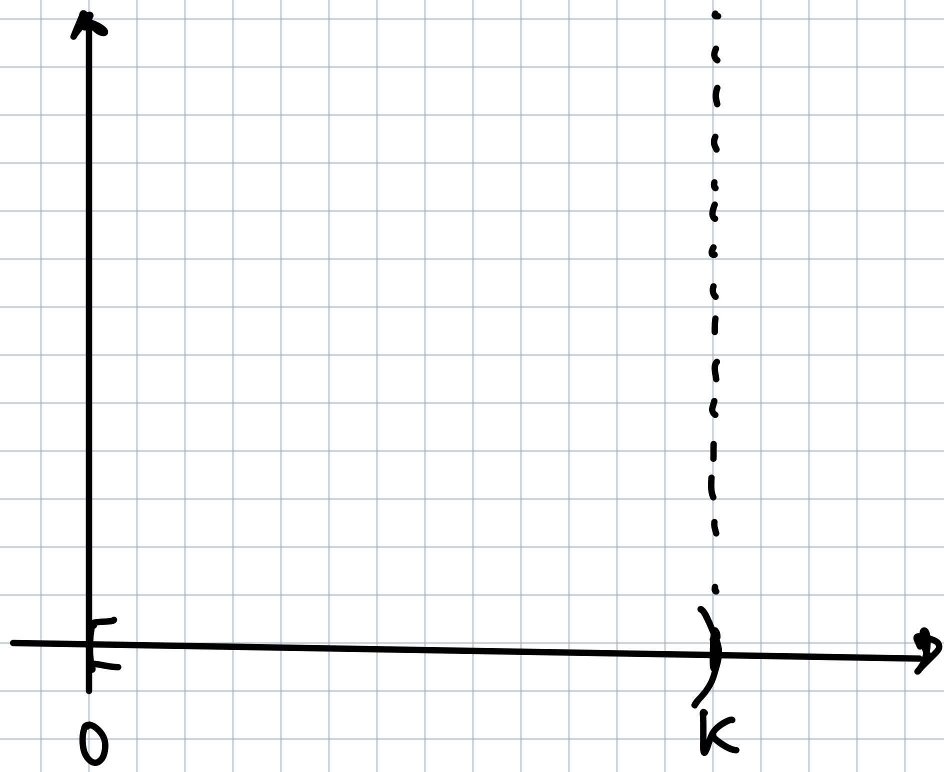
ISSUES

1. EXISTENCE
2. LOCAL VS GLOBAL SOLUTIONS
3. INTERIOR VS BOUNDARY SOLUTIONS (relative to the feasible set)

(1) An optimisation problem has a solution if the objective function is continuous and the feasible set is non-empty, closed and bounded.

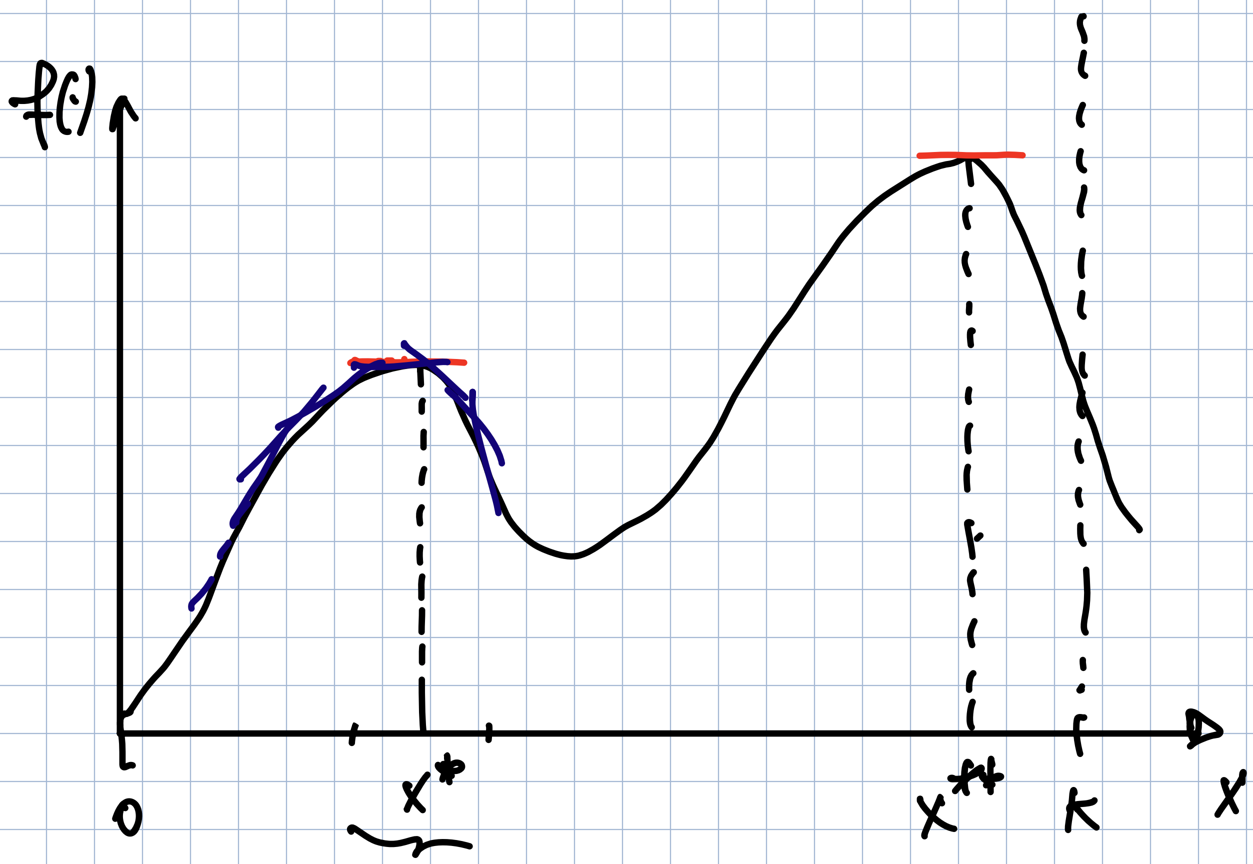


Every continuous function on a non-empty closed and bounded set has a max and min value



Construct examples that violate continuity of the obj. function or closedness or boundedness of the feasible set that violate Weierstrass theorem.

② LOCAL AND GLOBAL OPTIMIZERS



x^* is a local max since $f(x^*) \geq f(\tilde{x})$ $\forall\ \tilde{x} \in N_{x^*}$ (neighborhood of x^*).

The same is true for x^{**}.

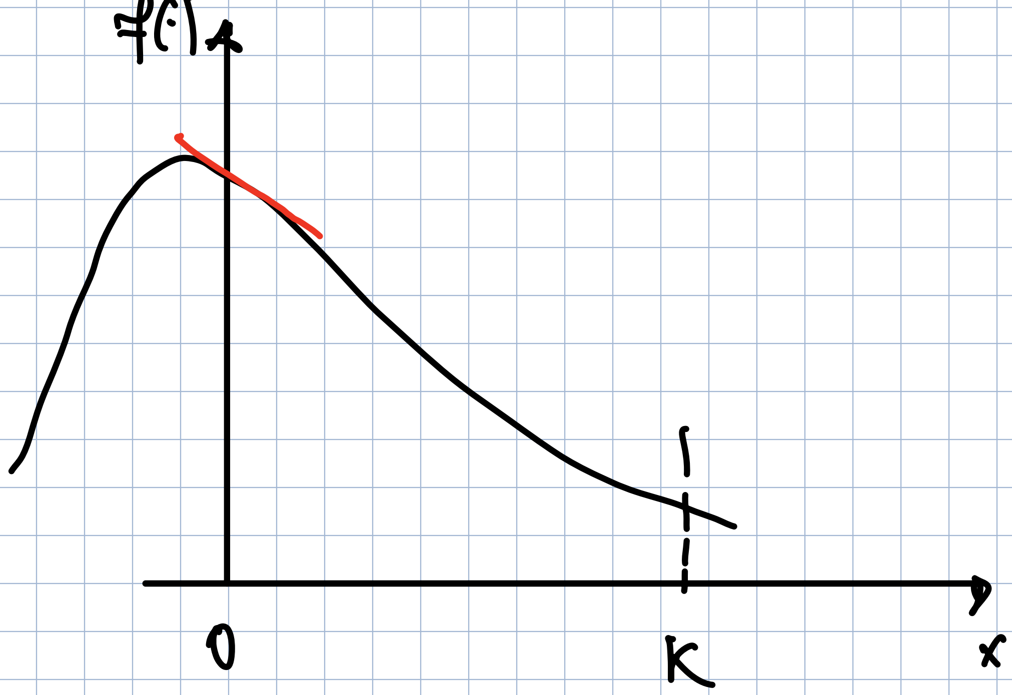
However, x^* is not a global max.

When searching for an optimizer, we use local approach. The characterization via first and second-order conditions identifies local maxima.

$f'(x^*) = 0$
and
$f'(\cdot)$ decreasing in x } necessary and suff. conditions for a max

NOTICE: if the optimizer is at the boundary of some non-negativity constraint (say $x \geq 0$) $\Rightarrow$ $f'(x) = 0$ will not hold at the optimum.

Ex.

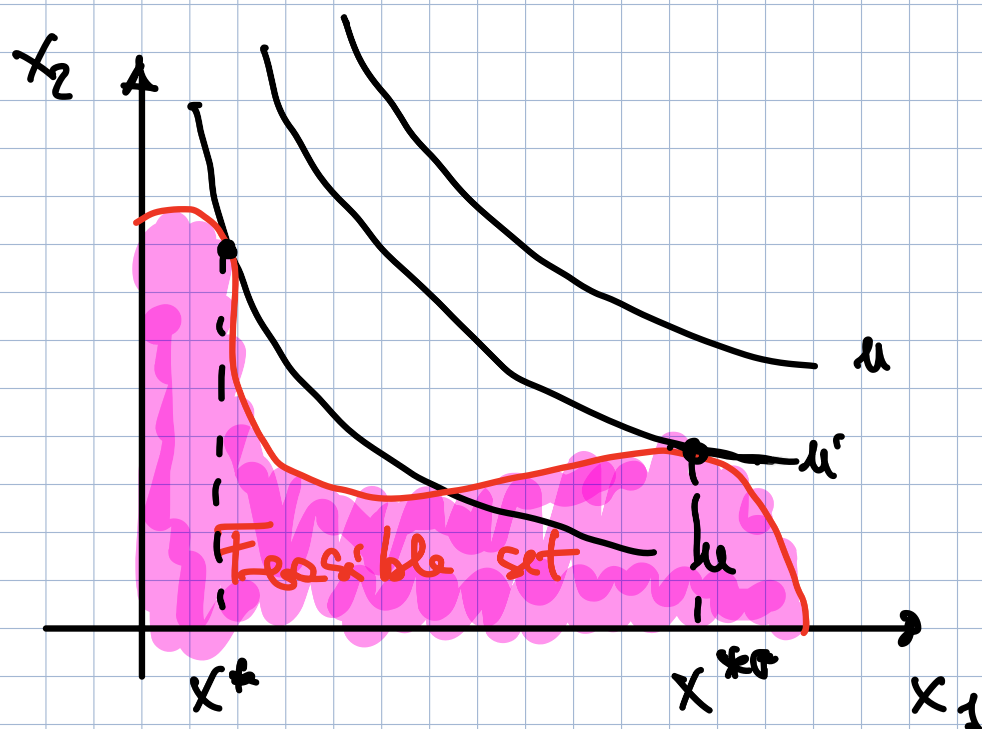


the FOC for corner solutions is:

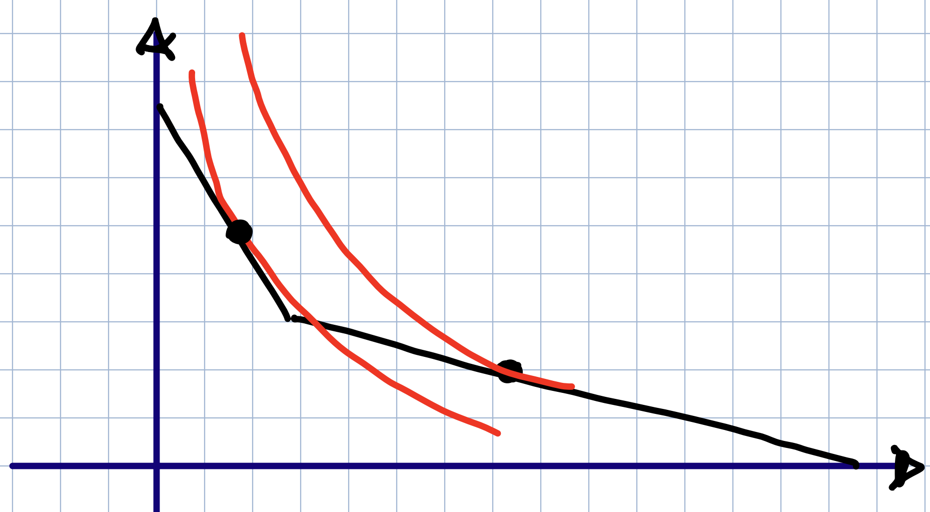
$$f'(x) \leq 0 \qquad x \geq 0 \qquad x \cdot f'(x) = 0$$

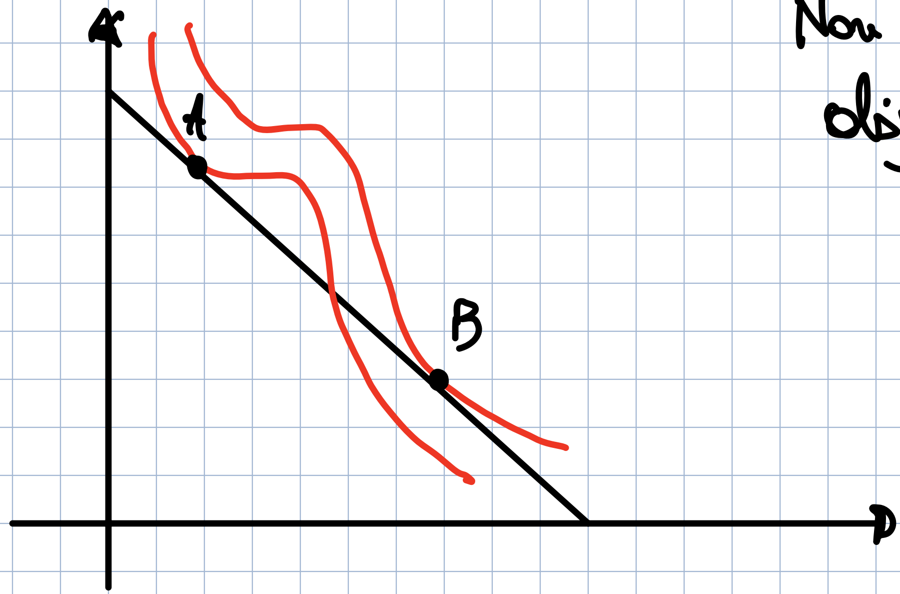
Which are the conditions that guarantee that every local optimum is also a global one?

Separating hyperplane theorem: Every local optimum is also a global one if the objective function is quasi-concave and the feasible set is a convex set.



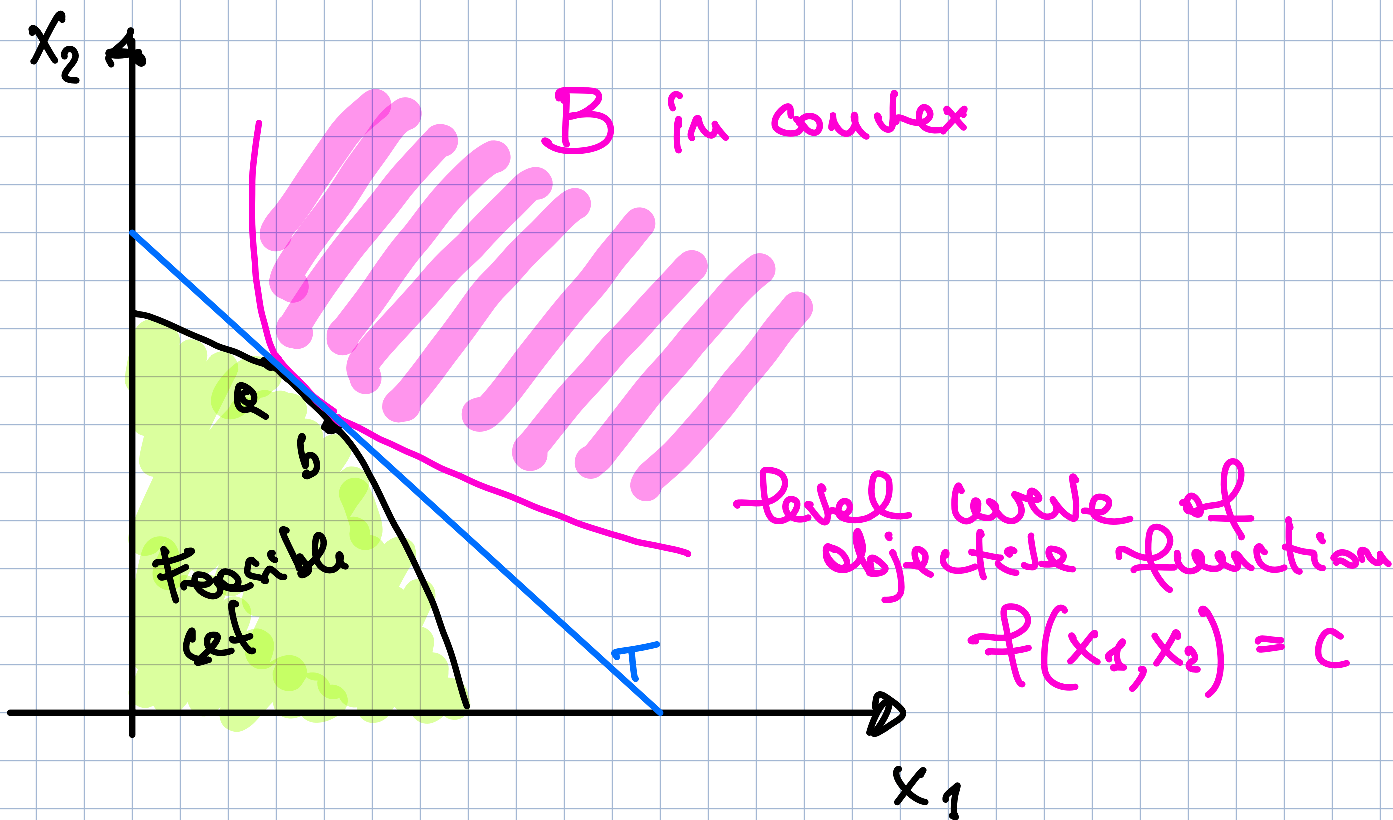
Non-convex feasible set.





Non quasi-concave obj. function. (verify !!)

If $L=2$ illustrating the separating hyperplane theorem



Let T be the line passing through
ab which "separates" the orthant
in two regions : one in which all bundles of
the feasible set lie, the other in which
points of the contour (pink) set are preferred to
these lie.

Such a line always exists if the feasible
set is convex and the objective function
is quasi-concave.