# MATHEMATICS
# Time allowed 1 Hour

Penalties: ☐ ☐

A.Y. 2025/2026 - Sample exam

## EXAM REGULATIONS.

**Any kind of electronic device (calculators, smartphones, smartwatches etc..) is FORBIDDEN. Put all your devices in the location indicated by the examiners. If an attendee violates this rule (even if the device is switched off or in offline mode), she/he will be immediately expelled from the exam session and her/his exam will be invalidated.**

***Examiners will mark only THIS A3 paper SHOWING THE UNIVERSITY LOGO on the upper left corner. However, any other additional papers (drafts, notes, scribbles or anything else) MUST be delivered to the examiners as a proof that you solved the problems by yourself.***

***WARNINGS: if, during the exam correction phase, the examiners should find a suspicion of cheating, the student can be summoned for an oral exam to verify the truthfulness of what has been written.***
***Make sure that you indicate all necessary steps to solve the exercises. If a justification of your result is missing the exercise will not be marked.***

***PENALTIES: Each examiner can, UNQUESTIONABLY, assign a penalty if she/he considers that two participants have communicated with each other (in any way). One penalty does not imply anything for the valuation of the exam. However, should an attendee be given two penalties, she/he will be immediately expelled from the exam session and her/his exam will be invalidated.***

***SUGGESTION: Remember to always double check the consistency of your results. EVEN IF CORRECT, inconsistent statements will result in a negative impact on the final grade of the exam. Use a clear and clean handwriting. Unclear or ambiguous sentences may result in a negative impact on the final grade of the exam.***

**By signing below you fully accept the exam regulations.**

MATRICOLA .................. Lastname ...................... Name ......................

The exam is divided in 2 parts. Part 1 (6 exercises) refers to questions from 1 to 6. This part amounts to 20 points. You must get at least 16 points in this part. Part 2 (2 exercises) refers to questions 7 and 8. This part amounts to 12 points. **Examiners will mark Part 2 only if you get 16 points or more in Part 1.** If you get less than 16 points in Part 1, you have not passed the exam. To pass the exam you must get at least 16 points in Part 1 AND 18 points or more in total.

**If you intend to withdraw please tick the box below and sign next.**

☐ Withdrawn Signature ..........................................................................

1. Compute the sum of the series $\sum_{n=0}^{\infty}\left(\frac{1-a}{a}\right)^n$. For which positive values of $a$ does the sum converge?

Indicate in short all steps to answer the question and the final answer in the space below.

This is a geometric series, so it converges for $\left|\frac{1-a}{a}\right| < 1$.

$\left|\frac{1-a}{a}\right| < 1 \Leftrightarrow -a < 1-a < a \Leftrightarrow 0 < 1 < 2a \Leftrightarrow \boxed{a > \frac{1}{2}}$.

2. Establish in which intervals of the real line the function $f(x) = 1 - ax^3$ is increasing/decreasing/constant. Provide the answer in terms of $a < 0$, $a = 0$, $a > 0$.

Indicate in short all steps to answer the question and the final answer in the space below.

$f'(x) = -3ax^2$, so considering different values of $a$:

- If $a > 0$, $f'(x) < 0$ $\forall a \neq 0$, so $\boxed{\text{always decreasing}}$
- If $a = 0$, $f'(x) = 0$ so $\boxed{\text{always constant}}$
- If $a < 0$, $f'(x) > 0$ $\forall a \neq 0$, so $\boxed{\text{always increasing}}$

3. Compute the limit $\lim_{x\to 2^+}\frac{1}{x-2}$ and $\lim_{x\to 2^-}\frac{1}{x-2}$. Does the limit $\lim_{x\to 2}\frac{1}{x-2}$ exist?

Indicate in short all steps to answer the question and the final answer in the space below.

$\frac{1}{x-2} > 0$ for $x > 2$ and $\frac{1}{x-2} < 0$ for $x < 2$, so $\boxed{\lim_{x\to 2^+}\frac{1}{x-2} = +\infty}$, $\boxed{\lim_{x\to 2^-}\frac{1}{x-2} = -\infty}$ and combining these two we see that $\boxed{\lim_{x\to 2}\frac{1}{x-2} \text{ does not exist}}$.

4. Compute all eigenvalues of the following matrix

$$\begin{pmatrix} 2 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Indicate in short all steps to answer the question and the final answer in the space below.

This is an upper triangular matrix;

$$\det\begin{pmatrix} 2-\lambda & 1 & -2 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda)(-\lambda) = 0 \text{ gives } \lambda = \boxed{2, 0, -1}$$

5. Compute the following indefinite integral

$$\int \frac{\log(x)}{2x}\,dx$$

Indicate in short all steps to answer the question and the final answer in the space below.

Substituting $y = \log x$, we have $dy = \frac{dx}{x}$ and so

$$\int \frac{\log(x)}{2x}dx = \frac{1}{2}\int y\,dy = \frac{1}{2}\frac{y^2}{2} + C = \boxed{\frac{1}{4}(\log(x))^2 + C}$$

6. Indicate the number of solutions of the following linear system as the parameter $k$ changes:

$$\begin{cases} x + y + kz &= 2 \\ x &= 1 \\ x - y + z &= 0 \end{cases}$$

Indicate in short all steps to answer the question and the final answer in the space below.

Using $x = 1$ to simplify the other equations gives

$$\begin{cases} x + kz &= 1 \\ -y + z &= -1, \end{cases}$$

and since $\det\begin{pmatrix} 1 & k \\ -1 & 1 \end{pmatrix} = 1 + k$ the system has $\boxed{\text{one solution for } k \neq -1}$. When $k = 1$, the two equations are the same except for a $-$ sign, so $\boxed{\text{for } k = -1 \text{ infinitely many solutions}}$,

7. Study the graph of the function

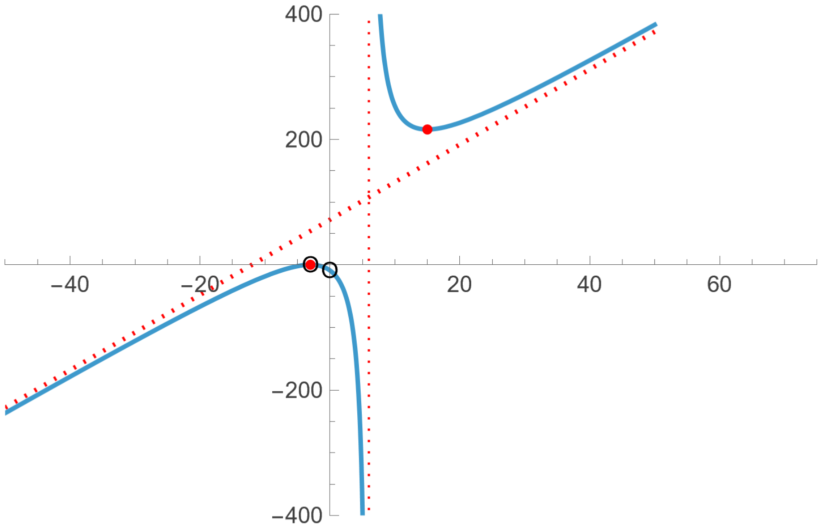
$$f(x) = \frac{6(x+3)^2}{x-6}$$

In particular: examine the domain, asymptotes, $x-$ and $y-$ intercepts, monotonicity, and maxima/minima, and sketch the graph of the function. **Important HINT**: omit 2nd order derivatives. Provide below all necessary steps to solve the exercise.

Answer:
domain: $\boxed{x \neq 6}$
intersection with x-axis: $f(x) = 0 \Rightarrow (x+3)^2 \Rightarrow x = -3$, so $\boxed{(-3, 0)}$
intersection with y-axis: $f(0) = \frac{6\dot{9}}{-6} = -9$, so $\boxed{(0, -9)}$
vertical asymptote at $x = 6$
horizontal asymptotes: none, $\lim_{x \to \pm\infty} f(x) = \pm\infty$
oblique asymptotes: $\frac{(x+3^2)}{x-6} = \frac{x^2+6x+9}{x-6} = x + \frac{x^2+6x+9-x^2+6x}{x-6} = x + \frac{12x+9}{x-6} = x + 12 - \frac{63}{x-6}$, so $f(x) = 6x + 72 - \frac{378}{x-6}$ and finally $\boxed{y = 6x + 72}$
monotonicity:

$$f'(x) = 6\frac{2(x+3)(x-6) - (x+3)^2}{(x-6)^2} = 6(x+3)\frac{1x - 12 - x - 3}{(x-6)^2} = 6(x+3)\frac{x-15}{(x-6)^2},$$

which is positive for $x < -3$ and $x > 15$, negative for $-3 < x < 6$ and $6 < x < 15$, zero for $x = -3, 15$, and undefined for $x = 6$. Putting this together, $f$ is $\boxed{\text{increasing on } (-\infty, -3) \text{ and } (15, \infty}$ and $\boxed{\text{decreasing on } (-3, 6)}$ and $(6, 15)$. maxima/minima: $f$ goes to $-\infty$ and $+\infty$, so it has no global (only local) extrema. Examining the intervals of monotonicity, we see that $f$ has a $\boxed{\text{local maximum at } x = -3}$ and a $\boxed{\text{local minimum at } x = 15}$.



8. Consider the function

$$f(x,y) = x^2y + 2y^2 + 16y$$

Answer the following questions:

(a) Compute the gradient and the Hessian matrix of the function.

(b) Determine the equation of the tangent plane at the point $(1,1)$

(c) Find all stationary points, if any.

(d) Determine when possible if they are local maxima/minima or saddle points.

Provide below all necessary steps to solve the exercise.

Answer:

(a) The partial derivatives are

$$\frac{\partial f}{\partial x}(x,y) = 2xy, \quad \frac{\partial f}{\partial y}(x,y) = x^2 + 4y + 16$$

and

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2y, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 2x, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 4;$$

assembling this

$$\boxed{\nabla f(x,y) = \begin{pmatrix} 2xy \\ x^2 + 4y + 16 \end{pmatrix}, \quad H_f(x,y) = \begin{pmatrix} 2y & 2x \\ 2x & 4 \end{pmatrix}}$$

(b) Evaluating the function and the gradient,

$$f(1,1) = 1 + 2 + 16 = 19, \quad \frac{\partial f}{\partial x}(1,1) = 2, \quad \frac{\partial f}{\partial y}(1,1) = 1 + 4 + 16 = 21;$$

then the equation of the tangent plane is $z = 19 + 21(x-1) + 21(y-1)$ or, simplifying,

$$\boxed{z = 2x + 21y - 4}.$$

(c) Solving $\frac{\partial f}{\partial x} = 0$ gives $x = 0$ and/or $y = 0$. When $y = 0$, $\frac{\partial f}{\partial y}(x,0) = x^2 + 16$, which cannot be zero. On the other hand setting $x = 0$, $\frac{\partial f}{\partial y}(0,y) = 4y + 16 = 0$ gives $y = -4$. Thus, the only stationary point is $\boxed{(0,-4)}$.

(d) At $(x,y) = (0,-4)$, the Hessian is $\begin{pmatrix} -8 & 0 \\ 0 & 4 \end{pmatrix}$, and since this is a diagonal matrix we see immediately that its eigenvalues are $4, -8$. Therefore the Hessian is indeterminate, and $\boxed{(0,-4) \text{ is a saddle point}}$.