Calculus, Bachelor in Global Governance.
2026/27 First semester (until October 7)

Yoh Tanimoto

Dipartimento di Matematica, Università di Roma Tor Vergata
Via della Ricerca Scientifica 1, I-00133 Roma, Italy
email: `hoyt@mat.uniroma2.it`

2026.09.14: Introduction of the course, Functions

Basic information

Lecture notes (or exercises): GG → Calculus → Teaching material,

or Yoh Tanimoto → Teaching → 2026 GG (From October 8, Prof. Cipolloni will give lessons)

Textbook: Hoffmann, Bradley, Sobecki, Price "Applied Calculus for Business, Economics, and the Social and Life Sciences"

Office hours: Wed. 14:00–15:00 (or by appointment)

Why study calculus?

You will study law, economics, history, English, logic, statistics, group interaction, etc. In some of these disciplines, numbers will play an important role.

Example: Tax rate and government revenue

Imagine that the government imposes a tax on business activities. If the tax rate is lower, there are more activities. If the tax rate is higher, there are fewer activities.

τ: tax rate.

$B(\tau)$: business activity at tax rate τ.

What happens to the tax revenue if τ rises? → Model.

τ	0	20	40	60	80	100
$B(\tau)$	100	80	60	40	20	0
tax revenue $\tau \cdot B(\tau)/100$	0	16	24	24	16	0

The tax revenue of the government is $B(\tau) \cdot \tau/100$.

A higher tax rate does not mean higher revenue.

Mathematical models help us understand what happens when we change something... if the models reflect the real world correctly.

Example: Minimum wage and employment rate

Let us consider what happens if the government imposes a minimum wage.

W: minimum wage.

$L(W)$: labour demand (employment).

What happens if W rises in one state, but not in another state?

Assume that the labour market is in "perfect competition." If the minimum wage were raised, companies would have less profit from workers with lower wage. So we expect less employment.

Data: New Jersey, 1992. \$4.25/h $\to$ \$5.05/h.[1] Slight increase in employment at fast-food restaurants. (Later study: small negative effect on employment in general.[2])

To understand how society works:

- need to make correct assumptions;
- need to understand how the model works;
- need to check whether the model matches reality.

In Calculus, we study tools to understand, use, and analyze models correctly: functions, derivatives, integrals, graphs, differential equations.

Functions (Textbook section 1.1)

A function is a rule that assigns to each number x in A ("domain") one (and only one) number y in B ("codomain", sometimes "range").

$$y = f(x).$$

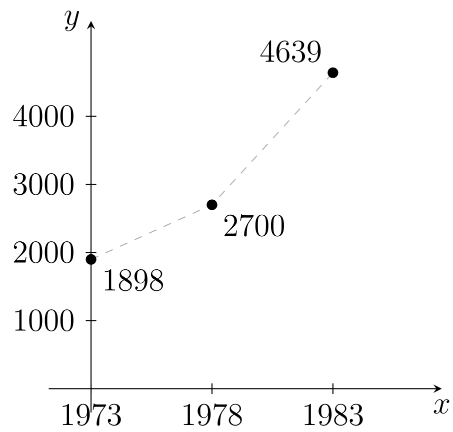
[1]D. Card and A. B. Krueger, "Minimum Wages and Employment: A Case Study of the Fast-Food Industry in New Jersey and Pennsylvania," *American Economic Review* **84** (1994), 772–793. nber.org/papers/w4509.

[2]D. Neumark and P. Shirley, "Myth or Measurement: What Does the New Minimum Wage Research Say about Minimum Wages and Job Loss in the United States?," *Industrial Relations* **61** (2022), 384–417. nber.org/papers/w28388.

An example from data: average tuition fees, 4-year college.

x (year)	1973	1978	1983
$f(x)$ (US dollar)	1898	2700	4639

The domain here is a finite set of years, so the graph is a set of separate points, not a curve.



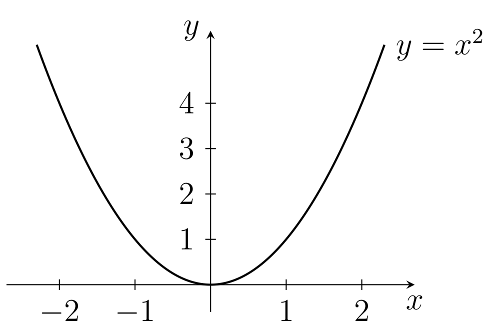
An example from a model:

$$y = f(x) = x^2.$$

If we take concrete numbers,

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2
$f(x)$	4	1	$\frac{1}{4}$	0	$\frac{1}{9}$	$\frac{1}{4}$	1	4

It graph can be drawn as follows.

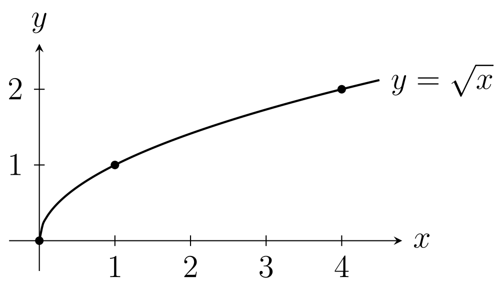


Examples

$$f(x) = \sqrt{x}$$

is defined for $x \geq 0$.

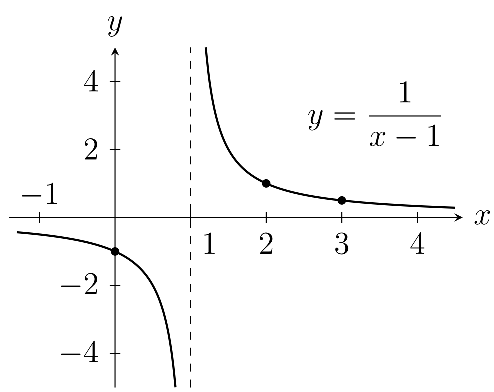
$$f(0) = 0, \qquad f(1) = 1, \qquad f(4) = 2.$$



$$f(x) = \frac{1}{x-1}$$

is defined for $x \neq 1$.

$$f(0) = -1, \qquad f(2) = 1, \qquad f(3) = \frac{1}{2}.$$

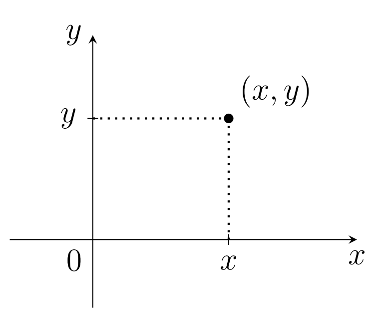


Graph of a function (section 1.2)

If we have a function f, we can plot its graph.

Take the example above.

In general, consider a (data) point (x, y), where x, y are numbers. We can plot that point on a plane. x, y are called the coordinates of the point (x, y).



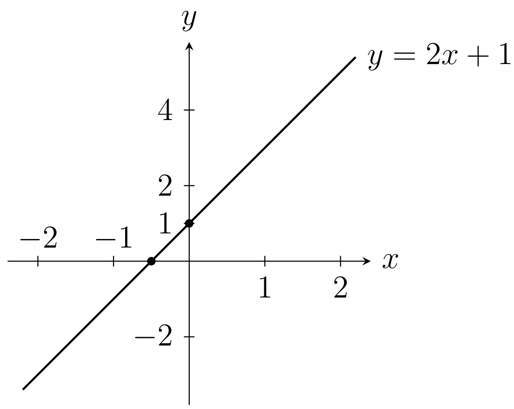
The graph of a function f is the collection of the points

$$(x, f(x)),$$

where x belongs to the domain of f.

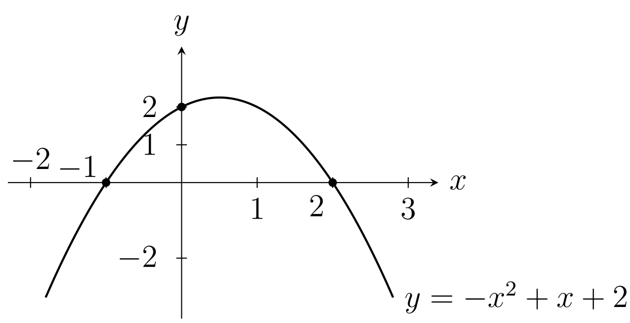
Example.

$$f(x) = 2x + 1.$$



Another sketch in the source:

$$f(x) = -x^2 + x + 2.$$



When does the graph intersect the x-axis and y-axis?

The **y-intercept** is $(0, f(0))$. The **x-intercept** is some $(x, f(x))$ such that $f(x) = 0$.

Example.

$$f(x) = 2x + 1.$$

y-intercept: $(0, 1)$.

x-intercept:

$$2x + 1 = 0 \quad \Longrightarrow \quad x = -\frac{1}{2},$$

so $(-\frac{1}{2}, 0)$.

Example.

$$f(x) = -x^2 + x + 2.$$

y-intercept: $(0, 2)$.

x-intercepts:

$$-x^2 + x + 2 = 0 \quad \iff \quad x^2 - x - 2 = 0 \quad \iff \quad (x+1)(x-2) = 0,$$

so $(-1, 0)$ and $(2, 0)$.

Example.

$$f(x) = \frac{x+1}{x-2}.$$

This is defined for $x \neq 2$.

y-intercept:

$$f(0) = \frac{0+1}{0-2} = -\frac{1}{2},$$

so $\left(0, -\frac{1}{2}\right)$.

x-intercept: a fraction is 0 exactly when its numerator is 0 (and its denominator is not), so

$$\frac{x+1}{x-2} = 0 \quad \iff \quad x + 1 = 0 \quad \iff \quad x = -1,$$

so $(-1, 0)$.

2026.09.15: Examples of functions

Linear function

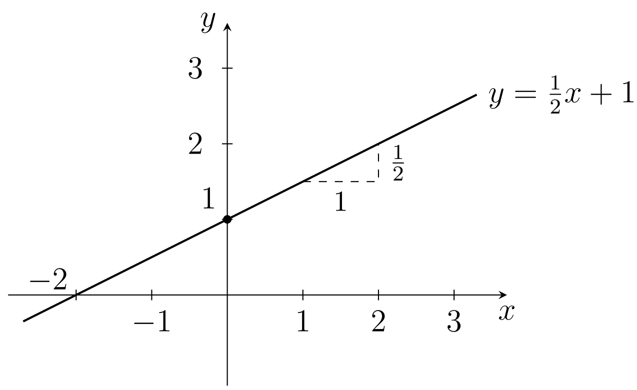
$$f(x) = ax + b,$$

here a, b are constants, which do not change when x changes.

Domain: all real numbers, $(-\infty, \infty)$.

The y-intercept is b; the slope is a.

The following is the graph of $f(x) = \frac{1}{2}x + 1$.



Quadratic function

Special forms:

$$f(x) = ax^2,$$

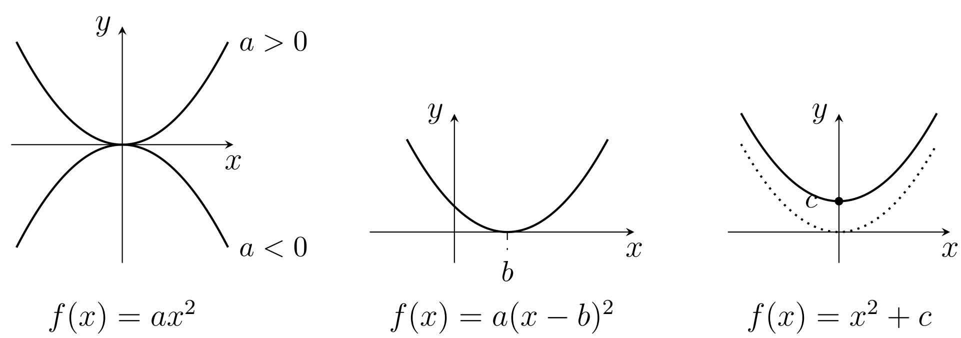
with sketches for $a > 0$ and $a < 0$. Domain: all real numbers, $(-\infty, \infty)$.

$$f(x) = a(x-b)^2,$$

with vertex at $x = b$;

$$f(x) = x^2 + c,$$

with vertical shift c.



In general,

$$f(x) = ax^2 + bx + c, \qquad a \neq 0.$$

If $a > 0$ or $a < 0$, it is a parabola.

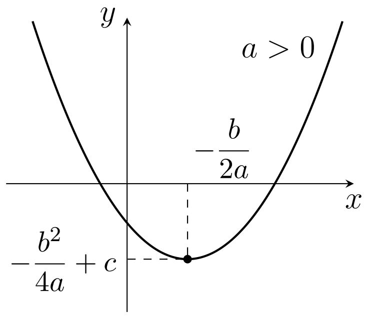
Completing the square:

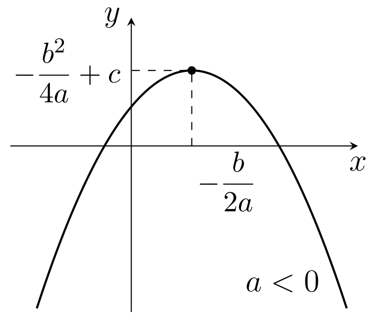
$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \end{aligned}$$

$$
\begin{aligned}
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.
\end{aligned}
$$

Recall:

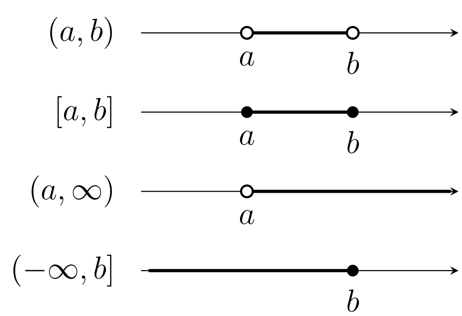
$$(x+A)^2 = x^2 + 2Ax + A^2.$$





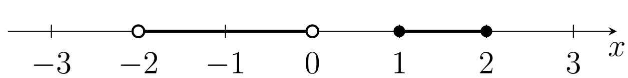
Intervals

$$
\begin{aligned}
(a,b) &: \text{ numbers larger than } a \text{ and smaller than } b, \\
[a,b] &: \text{ numbers not smaller than } a \text{ and not larger than } b, \\
(a,\infty) &: \text{ numbers larger than } a, \\
(-\infty,b] &: \text{ numbers not larger than } b.
\end{aligned}
$$



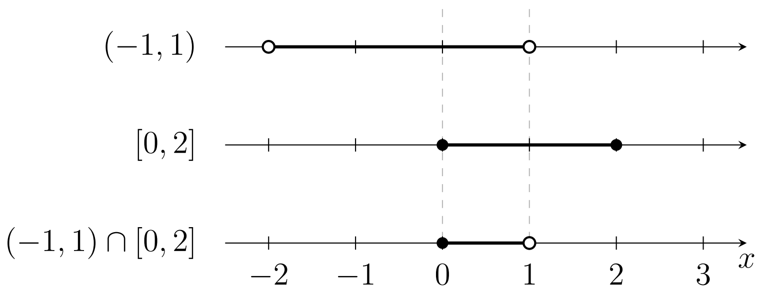
Union of intervals is the "set" of points contained either in one or the other:

$$(-2,0) \cup [1,2].$$



Intersection of intervals is the "set" of points contained in both:

$$(-1, 1) \cap [0, 2] = [0, 1).$$

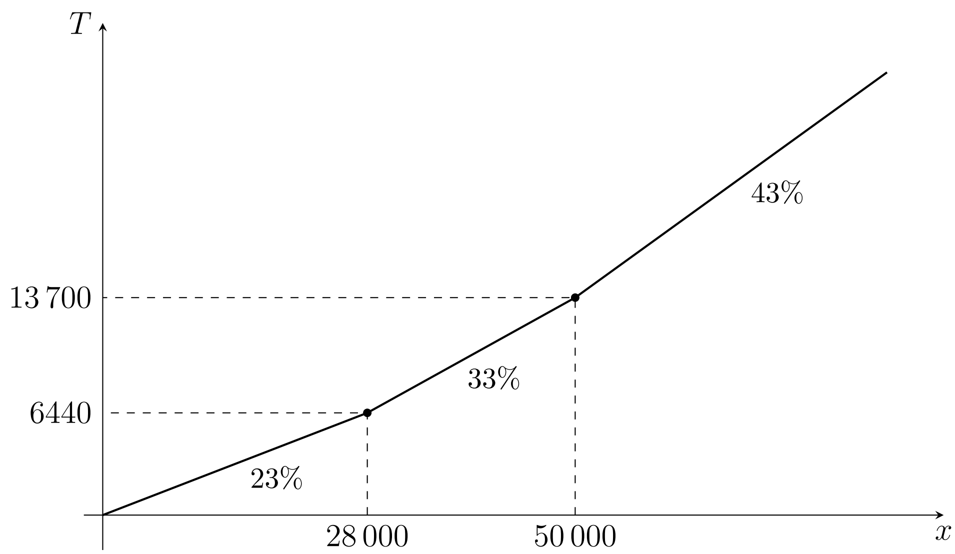


The left endpoint 0 is *included*, because 0 belongs to $[0, 2]$ and lies inside $(-2, 1)$. The right endpoint 1 is *excluded*, because 1 does not belong to $(-1, 1)$.

A real-world example: Italian income tax

Let x be the income (in euro). The income tax $T(x)$ is determined as follows: the rate is 23% on the part of the income up to 28 000, 33% on the part between 28 000 and 50 000, and 43% on the part above 50 000.[3]

$$T(x) = \begin{cases} 0.23\,x, & 0 \le x \le 28\,000, \\ 6440 + 0.33\,(x - 28\,000), & 28\,000 < x \le 50\,000, \\ 13\,700 + 0.43\,(x - 50\,000), & x > 50\,000. \end{cases}$$



[3] Agenzia delle Entrate, agenziaentrate.gov.it.

Operations on functions

If $f(x)$ and $g(x)$ are two functions, then:

- $f(x) + g(x)$ is another function;
- $f(x) - g(x)$ is another function;
- $f(x)g(x)$ is another function;
- $\dfrac{f(x)}{g(x)}$ is another function, defined for x such that $g(x) \neq 0$;
- $f(g(x))$ is another function, if $g(x)$ is in the domain of f (composition).

Examples.

- $f(x) = x$, $g(x) = a$:

$$f(x) + g(x) = x + a, \qquad f(x)g(x) = ax.$$

- $f(x) = x$, $g(x) = x^2$:

$$f(x) + g(x) = x + x^2, \qquad f(x)g(x) = x^3.$$

- $f(x) = x + 2$, $g(x) = x^2$:

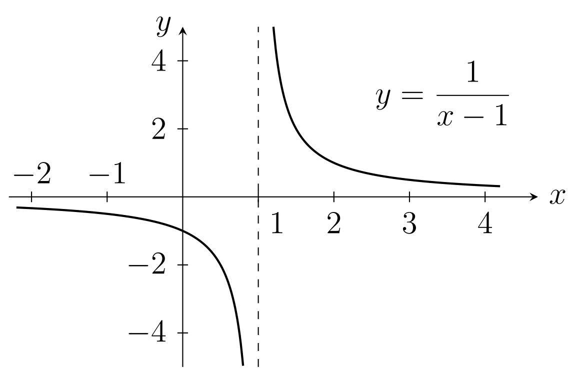
$$f(g(x)) = x^2 + 2, \qquad g(f(x)) = (x+2)^2.$$

Rational function

$$f(x) = \frac{1}{x-1}.$$

Domain:

$$(-\infty, 1) \cup (1, \infty).$$



Another rational function:

$$f(x) = \frac{x^3 + x^2 + 4}{x^2 - 3x}.$$

Root

For $x \geq 0$, there is y such that

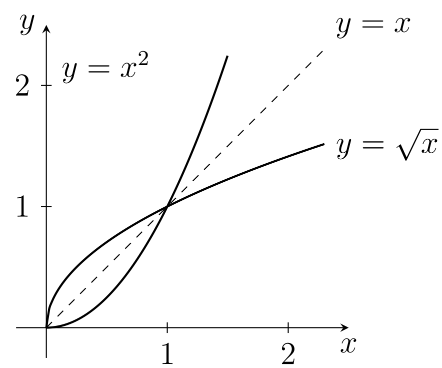
$$y^2 = x.$$

y is called the square root; we write $\sqrt{x}$.

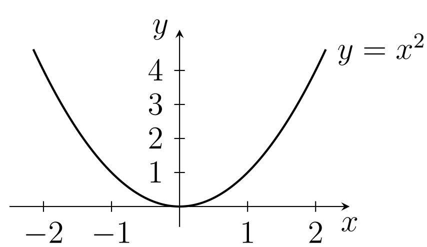
$$f(x) = \sqrt{x}$$

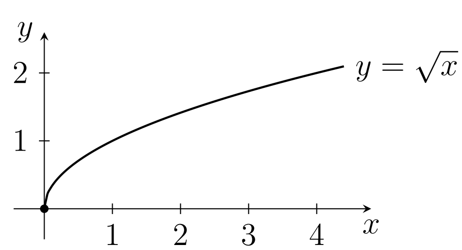
is a function defined on $[0, \infty)$.

$$\sqrt{1} = 1, \qquad \sqrt{0} = 0, \qquad \sqrt{4} = 2, \qquad \sqrt{9} = 3,$$
$$\sqrt{2} \approx 1.4142, \qquad \sqrt{3} \approx 1.73205\ldots$$



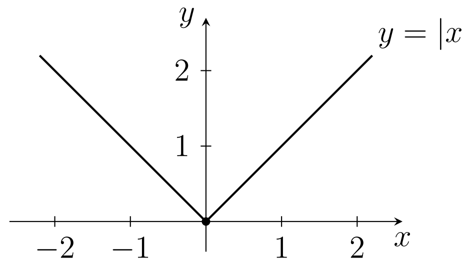
The graph of a function f is the collection of the points $(x, f(x))$, where x is in the domain of f.





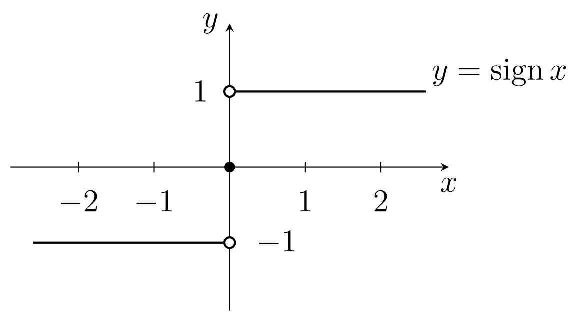
Absolute value

$$|x| = \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0. \end{cases}$$



Sign function

$$\operatorname{sign} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$



Exercises: domain of the following functions

1. $f(x) = 3x^4 + 4x$. Domain: $\mathbb{R}$.
2. $$f(x) = \frac{x^2}{x-1}.$$

   Need $x \neq 1$, so the domain is
   $$(-\infty, 1) \cup (1, \infty).$$
3. $$f(x) = \frac{x}{x^2-1}.$$

   Need
   $$x^2 - 1 \neq 0 \quad \Longleftrightarrow \quad (x-1)(x+1) \neq 0 \quad \Longleftrightarrow \quad x \neq -1, 1.$$

   Domain:
   $$(-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

4.

$$f(x) = \sqrt{x-1}.$$

Need

$$x - 1 \geq 0 \quad \iff \quad x \geq 1,$$

hence the domain is $[1, \infty)$.

2026.09.17: Limits

Limits of functions (Textbook section 1.5)

We want to study the behaviour of f, what happens when x approaches certain values.

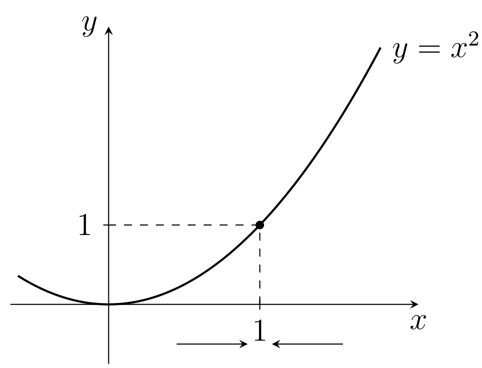
Examples. Consider

$$f(x) = x^2.$$

When x approaches 1, for example $1.1, 1.01, 1.001, \ldots$,

$$f(1.1) = 1.21, \qquad f(1.01) = 1.0201, \qquad f(1.001) = 1.002001,$$

while $f(1) = 1$.



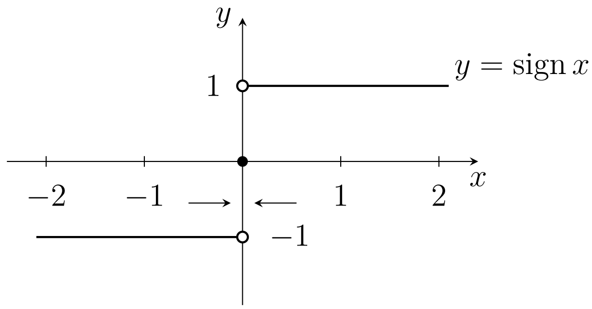
Consider

$$f(x) = \operatorname{sign} x.$$

When x approaches 0, for example $0.1, 0.01, 0.001, \ldots$,

$$\operatorname{sign}(0.1) = 1, \qquad \operatorname{sign}(0.01) = 1, \qquad \operatorname{sign}(0.001) = 1,$$

but $\operatorname{sign}(0) = 0$. For negative values approaching 0, the sign is -1.



Definition

If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c, from both sides, then we say that L is the limit of f as x approaches c, and write

$$\lim_{x \to c} f(x) = L.$$

Example.

$$f(x) = x^2, \qquad c = 1,$$

so

$$\lim_{x \to 1} x^2 = 1.$$

For $f(x) = \operatorname{sign} x$, when x approaches 0 from the right, $\operatorname{sign} x$ remains 1; from the left, it remains -1. Therefore

$$\lim_{x \to 0} \operatorname{sign} x$$

does not exist.

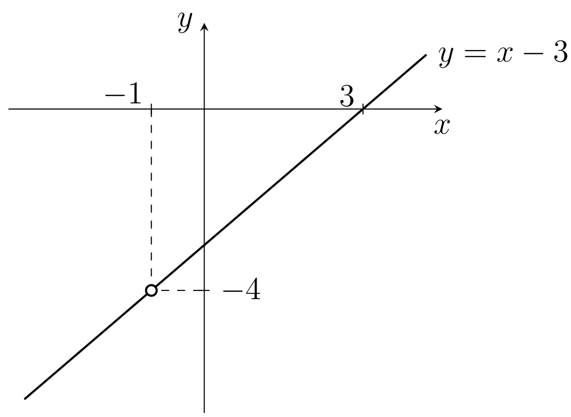
Example. Let

$$f(x) = \frac{x^2 - 2x - 3}{x + 1}, \qquad x \neq -1.$$

When x approaches -1,

$$f(x) = \frac{x^2 - 2x - 3}{x + 1} = \frac{(x+1)(x-3)}{x+1} = x - 3,$$

so $f(x)$ approaches -4.



The graph is the line $y = x - 3$ with the point $(-1, -4)$ removed: $f(-1)$ is not defined, but the height of the graph still approaches -4 as x approaches -1.

Example.

$$f(x) = \frac{1}{x}.$$

When x approaches 0 from the right, e.g. $0.1, 0.01, 0.001$,

$$f(0.1) = 10, \qquad f(0.01) = 100, \qquad f(0.001) = 1000.$$

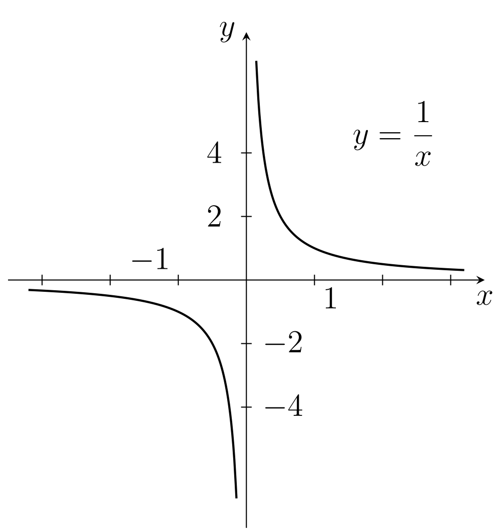
When x approaches 0 from the left, e.g. $-0.1, -0.01, -0.001$,

$$f(-0.1) = -10, \qquad f(-0.01) = -100, \qquad f(-0.001) = -1000.$$

So

$$\lim_{x \to 0} \frac{1}{x}$$

does not exist.



Geometrically, the limit exists when the height of the graph approaches L as x approaches c. This does not say much about $f(c)$ itself. Indeed, it can happen that

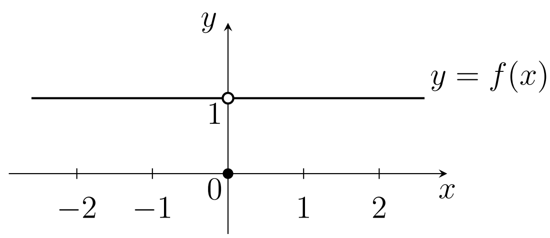
$$\lim_{x \to c} f(x) \neq f(c).$$

Example.

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = 1, \qquad \text{but} \qquad f(0) = 0.$$



Rules of limits

Quotient form

We can define

$$\frac{f(x)}{g(x)}$$

if $g(x) \neq 0$. What happens to the limit of the quotient?

If

$$\lim_{x \to c} g(x) \neq 0,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}.$$

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, the value $\lim_{x \to c} \frac{f(x)}{g(x)}$ can vary. This case is called **indeterminate**.

For example, $\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$ but $\lim_{x \to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x}$ does not exist.

Rules of limits

Assume that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Then

$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x),$$

$$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x),$$

$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right),$$

and

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}, \quad \text{if } \lim_{x\to c} g(x) \neq 0.$$

Examples.

$$f(x) = x.$$

When x approaches c, $f(x) = x$ approaches c.

$$g(x) = k \quad \text{(constant)}.$$

When x approaches c, $g(x) = k$ approaches k.

Example.

$$\lim_{x\to c} x^2 = \lim_{x\to c}(x\cdot x) = \left(\lim_{x\to c} x\right)\left(\lim_{x\to c} x\right) = c^2.$$

Also,

$$\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x),$$

and

$$\lim_{x\to c}(x^2+2x) = c^2+2c.$$

Example: rationalization

$$f(x) = \frac{\sqrt{x}-1}{x-1}.$$

When $x\to 1$, $x-1\to 0$. Numerically, values are close to 0.5. Indeed,

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x})^2-1^2}\\ &= \lim_{x\to 1}\frac{\sqrt{x}-1}{(\sqrt{x}+1)(\sqrt{x}-1)}\\ &= \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{1+1} = \frac{1}{2}.\end{aligned}$$

For an indeterminate limit, we cannot simply substitute x by c.

Example.

$$\lim_{x \to 1} \frac{3x^3 - 8}{x - 2}.$$

Since

$$\lim_{x \to 1} (x - 2) = -1 \neq 0,$$

we can substitute:

$$\lim_{x \to 1} \frac{3x^3 - 8}{x - 2} = \frac{-5}{-1} = 5.$$

Example.

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 3x + 2}.$$

Since the denominator tends to 0, this is indeterminate. Factor:

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x^2 - 3x + 2} &= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} \\ &= \lim_{x \to 1} \frac{x+1}{x-2} = \frac{2}{-1} = -2. \end{aligned}$$

If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

then f is called a polynomial, and

$$\lim_{x \to c} f(x) = f(c).$$

If

$$f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0},$$

then f is called a rational function, and

$$\lim_{x \to c} f(x) = f(c)$$

provided the denominator at c is nonzero.

2026.09.21: Infinite limits, one-sided limit

We studied the limit

$$\lim_{x \to c} f(x) = L,$$

meaning that $f(x)$ approaches L when x approaches c.

There are other cases, where f increases or decreases without bound. We say

$$\lim_{x \to c} f(x) = +\infty$$

if $f(x)$ increases without bound, and

$$\lim_{x \to c} f(x) = -\infty$$

if $f(x)$ decreases without bound.

Examples.

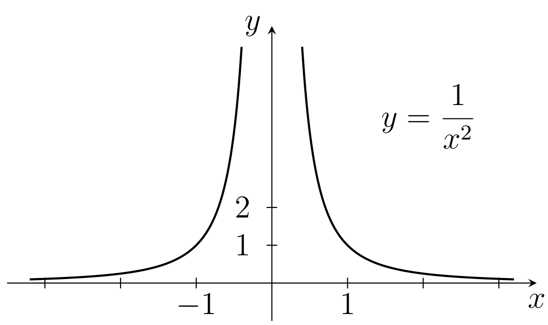
$$f(x) = \frac{1}{x^2}.$$

For instance,

$$f(0.1) = 100, \qquad f(0.01) = 10000, \qquad f(-0.001) = 1000000,$$

so

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$



Here both sides behave in the same way, so the limit is $+\infty$.

For

$$f(x) = \frac{1}{x},$$

positive values near 0 become large and positive, while negative values near 0 become large and negative, so

$$\lim_{x \to 0} \frac{1}{x}$$

does not exist.

If the values of f approach L as x increases without bound, we write

$$\lim_{x \to \infty} f(x) = L.$$

Similarly one may consider $x \to -\infty$.

Example.

$$f(x) = \frac{1}{x}.$$

$$f(10) = 0.1, \qquad f(100) = 0.01,$$

and

$$f(-10) = -0.1, \qquad f(-100) = -0.01.$$

Thus

$$\lim_{x\to\infty} \frac{1}{x} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Geometrically,

$$\lim_{x\to\infty} f(x) = L$$

exists if the graph of $f(x)$ approaches the line $y = L$, called a horizontal asymptote.

Similarly, if $k > 0$,

$$\lim_{x\to\infty} \frac{A}{x^k} = 0, \qquad \lim_{x\to-\infty} \frac{A}{x^k} = 0,$$

where A is a constant.

Example. Find

$$\lim_{x\to\infty} \frac{x^2}{1 + x + 2x^2}.$$

Directly this looks like the indeterminate form ∞/∞. Divide numerator and denominator by x^2:

$$\begin{aligned} \lim_{x\to\infty} \frac{x^2}{1 + x + 2x^2} &= \lim_{x\to\infty} \frac{1}{\frac{1}{x^2} + \frac{1}{x} + 2} \\ &= \frac{1}{2}. \end{aligned}$$

Algebraic limit rules also hold for $x \to \infty$ (when the relevant limits exist): sums, products, and quotients.

- $\lim_{x\to\infty}(f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$
- $\lim_{x\to\infty}(f(x) - g(x)) = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$
- $\lim_{x\to\infty}(f(x) \cdot g(x)) = (\lim_{x\to\infty} f(x)) \cdot (\lim_{x\to\infty} g(x))$
- $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ (if $\lim_{x\to\infty} g(x) \neq 0$).

Exercise:

$$\begin{aligned} \lim_{x\to\infty} \frac{2x^2 + 3x + 1}{3x^2 - 5x + 2} &= \lim_{x\to\infty} \frac{2 + \frac{3}{x} + \frac{1}{x^2}}{3 - \frac{5}{x} + \frac{2}{x^2}} \\ &= \frac{2}{3}. \end{aligned}$$

Example: crop yield model

If a crop is planted in soil where the nitrogen level is N, the crop yield can be "modelled" (known by experience) by

$$Y(N) = \frac{AN}{B+N},$$

where A and B are positive constants (Michaelis–Menten).

$$\lim_{N\to\infty} Y(N) = \lim_{N\to\infty} \frac{A}{\frac{B}{N}+1} = A.$$

Even if the nitrogen level increases, the crop yield does not increase unlimitedly.

Infinite-limit examples

$$f(x) = x, \qquad \lim_{x\to\infty} x = \infty, \qquad \lim_{x\to-\infty} x = -\infty.$$

$$f(x) = x^2, \qquad \lim_{x\to\infty} x^2 = \infty, \qquad \lim_{x\to-\infty} x^2 = \infty.$$

Exercise:

$$\lim_{x\to\infty} \frac{-x^3+2x+1}{x-3}.$$

This is an "indeterminate" form $-\infty/\infty$. We calculate

$$\begin{aligned}\lim_{x\to\infty} \frac{-x^3+2x+1}{x-3} &= \lim_{x\to\infty} -\frac{x^2-2-\frac{1}{x}}{1-\frac{3}{x}} \\ &= -\frac{\infty}{1} = -\infty.\end{aligned}$$

One-sided limits

Consider

$$\operatorname{sign} x = \begin{cases} 1, & x>0, \\ 0, & x=0, \\ -1, & x<0. \end{cases}$$

$$\lim_{x\to 0} \operatorname{sign} x$$

does not exist, but when x approaches 0 from the right, $\operatorname{sign} x$ remains 1. We write

$$\lim_{x\to 0^+} \operatorname{sign} x = 1.$$

More generally, if $f(x)$ approaches L when x approaches c from the right $(x > c)$, we write

$$\lim_{x \to c^+} f(x) = L.$$

For approach from the left $(x < c)$, we write

$$\lim_{x \to c^-} f(x) = L.$$

The two-sided limit exists if and only if both the one-sided limits exist.

Example. Tuition discount on the tuition fee at the University of Roma Tor Vergata depends on the ISEE x of their household (an indicator of income and assets). [4]

ISEE x	discount $f(x)$
$x < 26\,000$	100%
$26\,000 \leq x \leq 28\,000$	60%
$28\,000 < x \leq 30\,000$	55%
$x > 30\,000$	0%

$\lim_{x \to 26000^+} f(x) = 60, \lim_{x \to 26000^-} f(x) = 100.$

Example.

$$f(x) = \frac{1}{x}.$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Example.

$$\lim_{x \to 4^+} \frac{x-2}{x-4}.$$

Here

$$\lim_{x \to 4^+} (x-2) = 2, \qquad \lim_{x \to 4^+} \frac{1}{x-4} = \infty.$$

Thus

$$\lim_{x \to 4^+} \frac{x-2}{x-4} = \infty.$$

Exercises: Calculate

$$\lim_{x \to \infty} \frac{x^3 - 2x - 3}{x^4 + 5x},$$

$$\lim_{x \to 0^-} \frac{x^3 + 2}{x}.$$

[4] Exemptions for a.a. 2024/25: studenti.uniroma2.it. Ordinary taxation

Integer part

The integer part is written $\lfloor x \rfloor$ in the notes (i.e. the floor function).

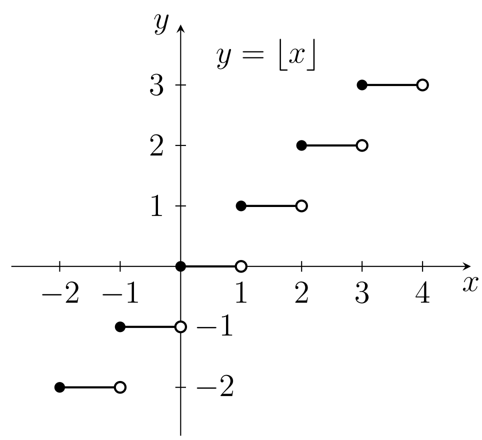
If $x = 1.4$, then

$$\lfloor x \rfloor = 1.$$

If $x = 3.14$, then

$$\lfloor 3.14 \rfloor = 3.$$

On each interval $[n, n+1)$ the value is constantly n, so the graph is a staircase: the left endpoint of each step belongs to the graph, the right one does not.



At $x = 1$,

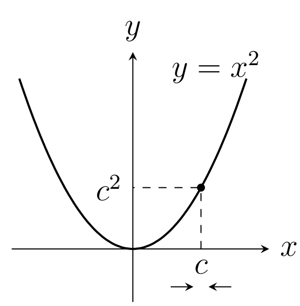
$$\lim_{x \to 1^+} \lfloor x \rfloor = 1, \qquad \lim_{x \to 1^-} \lfloor x \rfloor = 0.$$

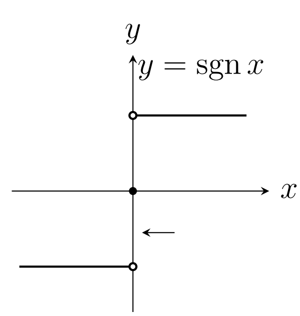
2026.09.22: Continuity

Continuity (Textbook section 1.6)

We have seen some functions and limits, for instance

$$\lim_{x \to c} x^2 = c^2, \qquad \lim_{x \to 0^+} \operatorname{sign} x = 1 \neq 0.$$





Definition. A function f is said to be *continuous* at c if

$$\lim_{x\to c} f(x) = f(c),$$

that is, c is in the domain of f, $\lim_{x\to c} f(x)$ exists, and it equals $f(c)$. Otherwise f is said to be *discontinuous* at c. The idea is that the graph of f is connected (unbroken) at c.

Examples.

- $f(x) = x$ is continuous: $\lim_{x\to c} x = c = f(c)$.
- $f(x) = x^2$ is continuous:
$$\lim_{x\to c} x^2 = \left(\lim_{x\to c} x\right)\left(\lim_{x\to c} x\right) = c \cdot c = c^2 = f(c).$$
- $f(x) = x^2 + x$ is continuous, using the rules of limits:
$$\lim_{x\to c}(x^2 + x) = \lim_{x\to c} x^2 + \lim_{x\to c} x = c^2 + c = f(c).$$
- $f(x) = \dfrac{1}{x}$. If $c \neq 0$, $\dfrac{1}{x}$ is continuous at c:
$$\lim_{x\to c} \frac{1}{x} = \frac{1}{\lim_{x\to c} x} = \frac{1}{c} = f(c).$$
If $c = 0$, $\dfrac{1}{x}$ is discontinuous at 0, because $\displaystyle\lim_{x\to 0} \frac{1}{x}$ does not exist.

These examples are all instances of the same principle: each rule of limits gives a rule of continuity.

Rules of continuity. Let f and g be continuous at c. Then

- $f(x) + g(x)$ and $f(x) - g(x)$ are continuous at c;
- $f(x)g(x)$ is continuous at c;
- $\dfrac{f(x)}{g(x)}$ is continuous at c, provided that $g(c) \neq 0$;
- if h is continuous at $f(c)$, then $h(f(x))$ is continuous at c (composition).

Each of these follows from the corresponding rule of limits. For the product, for instance,

$$\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right) = f(c)g(c),$$

which is exactly the statement that fg is continuous at c.

In general, a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

(a polynomial) is continuous at every c. A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials (a rational function), is continuous at every c such that $q(c) \neq 0$.

Example.

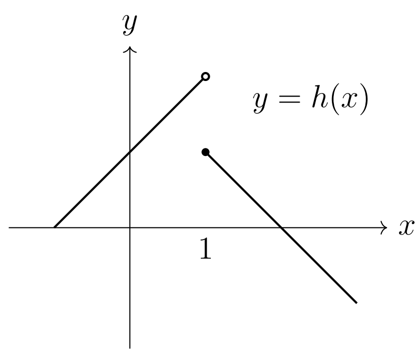
$$f(x) = \frac{x-1}{x-2}.$$

This f is continuous at $x = 3$. Indeed, $\lim_{x\to 3}(x-2) = 3-2 = 1 \neq 0$. Thus

$$\lim_{x\to 3} \frac{x-1}{x-2} = \frac{3-1}{3-2} = \frac{2}{1} = 2 = f(3).$$

Example.

$$h(x) = \begin{cases} x+1, & x < 1, \\ 2-x, & x \geq 1. \end{cases}$$



For $c \neq 1$, either $c < 1$ or $c \geq 1$. If $c < 1$, then $\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c}(x+1) = c+1 = h(c)$. If $c > 1$, then $\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c}(2-x) = 2-c = h(c)$.

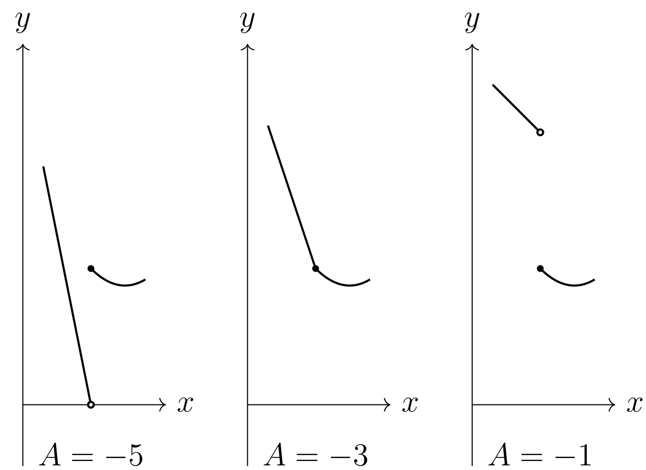
At $c = 1$,

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-}(x+1) = 1+1 = 2, \qquad \lim_{x\to 1^+} h(x) = \lim_{x\to 1^+}(2-x) = 2-1 = 1.$$

Since these one-sided limits disagree, $\lim\limits_{x\to 1} h(x)$ does not exist, so h is discontinuous at 1.

Exercise. For what value of A is the following f continuous for all x?

$$f(x) = \begin{cases} Ax+5, & x < 1, \\ x^2 - 3x + 4, & x \geq 1. \end{cases}$$



The picture above shows three different values of A: for $A = -5$ or $A = -1$ the graph jumps at $x = 1$, while for $A = -3$ the two pieces meet. f is already continuous for $x < 1$ and for $x > 1$. As for $x = 1$,

$$\lim_{x \to 1^-} (Ax + 5) = A + 5, \qquad \lim_{x \to 1^+} (x^2 - 3x + 4) = 1^2 - 3 \cdot 1 + 4 = 2.$$

For f to be continuous at 1, these must agree: $A + 5 = 2$, so $A = -3$.

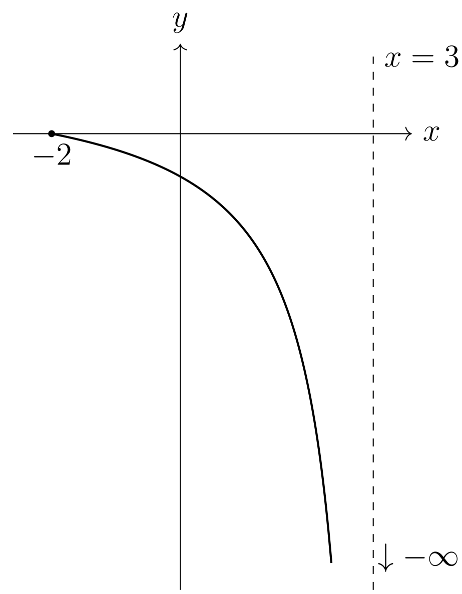
Continuity on a closed interval

For a function f defined on $[a, b]$, f is continuous on $[a, b]$ if it is continuous at each point of (a, b) and

$$f(a) = \lim_{x \to a^+} f(x), \qquad f(b) = \lim_{x \to b^-} f(x).$$

Example.

$$f(x) = \frac{x + 2}{x - 3} \qquad \text{on } (-2, 3).$$



f is a rational function, so it is continuous on $(-2, 3)$, but f is not continuous on $[-2, 3]$: $f(3)$ is not defined, and in fact

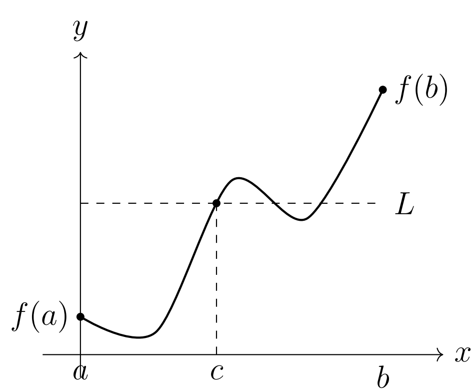
$$\lim_{x \to 3^-} \frac{x+2}{x-3} = -\infty.$$

Intermediate value theorem

Let f be a continuous function on $[a, b]$, and let L be a number between $f(a)$ and $f(b)$, that is,

$$f(a) < L < f(b) \quad \text{if } f(a) < f(b), \qquad f(b) < L < f(a) \quad \text{if } f(b) < f(a).$$

Then there is $c \in (a, b)$ such that $f(c) = L$.



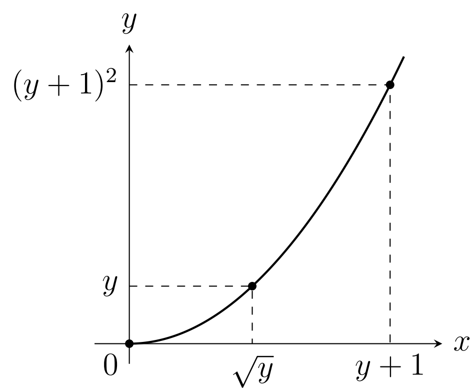
This is because the graph of f is “connected”: it rises from $f(a)$ to $f(b)$, so at some point c it must cross the height L. If you pour water into a bottle, at some point the level of the water reaches every height between the bottom and the top of the bottle.

Example. For $y > 0$, there is $x \geq 0$ such that $x^2 = y$.

Indeed, let $b = y + 1$. The function $f(x) = x^2$ is continuous on $[0, b]$, and

$$f(0) = 0 < y < (y+1)^2 = f(b).$$

By the intermediate value theorem, there is $x \in (0, b)$ such that $x^2 = y$, that is, $x = \sqrt{y}$.



Exercise. Show that the equation

$$x^3 - x - 1 = 0$$

has a solution c with $1 < c < 2$.

Solution. Let $f(x) = x^3 - x - 1$. Since f is a polynomial, it is continuous on $[1, 2]$.

$$f(1) = 1 - 1 - 1 = -1 < 0, \qquad f(2) = 8 - 2 - 1 = 5 > 0.$$

So $f(1) < 0 < f(2)$. By the intermediate value theorem (with $L = 0$), there is $c \in (1, 2)$ such that $f(c) = 0$, that is,

$$c^3 - c - 1 = 0.$$

2026.09.23: Exercises on limits and functions

Linear functions

$$f(x) = ax + b.$$

Exercises.

1. Find a linear function whose graph passes through $(2, -3)$ and $(0, 4)$.
2. Find a linear function whose graph passes through $(-2, 2)$ and $(2, 5)$.
3. Determine the x- and y-intercepts of

$$y = 3x - 6, \qquad y = \tfrac{1}{2}x + 5, \qquad y = x^2 - 3x + 2, \qquad y = x^3 - x.$$

4. Find a linear function with slope $-\frac{1}{2}$ whose graph passes through $(5, -2)$.
5. A manufacturer's total cost consists of a fixed overhead of \$5,000 plus production costs of \$60 per unit. Write the total cost as a function of the number of units.

Exercises: determine the limit

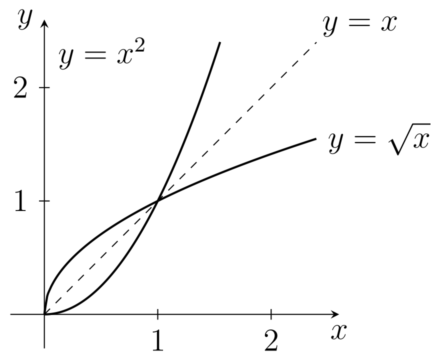
1. $\lim\limits_{x\to 2}\left(3x^2 - 5x + 2\right)$
2. $\lim\limits_{x\to 3}(x-1)^2(x+1)$
3. $\lim\limits_{x\to \frac{1}{3}} \dfrac{x+1}{x+2}$
4. $\lim\limits_{x\to 5} \dfrac{x+3}{5-x}$
5. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$
6. $\lim\limits_{x\to 5} \dfrac{x^2-3x-10}{x-5}$
7. $\lim\limits_{x\to 0} \dfrac{x(x^2-1)}{x^2}$
8. $\lim\limits_{x\to -2} \dfrac{x^2-x-6}{x^2+3x+2}$
9. $\lim\limits_{x\to 2^-} \dfrac{x^2+4}{x-2}$
10. $\lim\limits_{x\to 1} \dfrac{x-\sqrt{x}}{\sqrt{x}-1}$
11. $\lim\limits_{x\to \infty} \dfrac{x^3+2x+1}{10x-1}$
12. $\lim\limits_{x\to -\infty} \dfrac{x^3+5x^2}{-x^2+3}$

Exercises: determine the domain of the function

1. $f(x) = \sqrt{1-x}$
2. $f(x) = \sqrt{x^2-5x+4}$
3. $f(x) = \dfrac{1}{\sqrt{x^2-5x+4}}$

Power function

For $x \geq 0$, the equation $\sqrt{x} = y$ is equivalent to $x = y^2$. Therefore the graph of $f(x) = \sqrt{x}$ is obtained by reflecting the graph of $g(x) = x^2$ along the line $y = x$.



The graph of $f(x) = \sqrt{x}$ is connected, so $\sqrt{x}$ is continuous.

Similarly, we can define

$$\sqrt[3]{x} = x^{\frac{1}{3}} \quad \text{(cubic root)},$$

and more generally the power function x^a for real a. They are continuous on $[0, \infty)$.

2026.09.24: Exponential functions

Powers with a rational exponent

Let us recall that, for a number x,

$$x^1 = x, \qquad x^2 = x \cdot x, \qquad x^3 = x \cdot x \cdot x, \qquad \ldots$$

and hence, for example,

$$x^2 \cdot x^3 = x^5, \qquad (x^2)^3 = x^6,$$

and more generally $x^a x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$ for positive integers a and b.

Moreover, for $x > 0$ we saw that there is the square root $\sqrt{x}$, with $(\sqrt{x})^2 = x$. We write

$$\sqrt{x} = x^{\frac{1}{2}}$$

because $x = x^1 = \left(x^{\frac{1}{2}}\right)^2$. Similarly, the n-th root is denoted by $x^{\frac{1}{n}}$.

In general, for a rational number $\frac{p}{q}$ (such as $\frac{1}{2}$, $\frac{5}{3}$), we write

$$x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = (x^p)^{\frac{1}{q}}.$$

For a negative exponent $-\frac{p}{q}$ we define, still for $x > 0$,

$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}}, \qquad \text{and} \qquad x^0 = 1.$$

Examples.

$$4^{\frac{1}{2}} = \sqrt{4} = 2, \qquad 4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{\left(4^{\frac{1}{2}}\right)^3} = \frac{1}{2^3} = \frac{1}{8}, \qquad 27^{-\frac{2}{3}} = \frac{1}{9}.$$

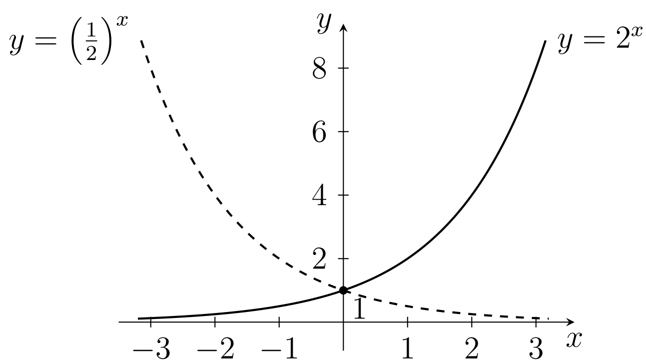
The exponential function

Now let $a > 0$ be constant. We can consider the function

$$f(x) = a^x,$$

first defined for rational numbers x, but we can define a^x for all real numbers x by continuity.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Exponential rules

Let $a > 0$, $b > 0$ and $x, y \in \mathbb{R}$ (real numbers). Then

- if $b \neq 1$, then $b^x = b^y$ if and only if $x = y$;
- $b^x b^y = b^{x+y}$;
- $\frac{b^x}{b^y} = b^{x-y}$;
- $(b^x)^y = b^{xy}$;
- $(ab)^x = a^x b^x$;
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

Examples.

$$3^2 \cdot 3^3 = 3^5 = 243, \qquad \left(2^3\right)^2 = 2^6 = 64, \qquad \frac{2^3}{2^5} = 2^{-2} = \frac{1}{4}.$$

Properties of exponential functions

Let $b > 0$, $b \neq 1$.

- $f(x) = b^x$ is continuous, and $b^x > 0$;
- $b^0 = 1$;
- if $b > 1$, then b^x is increasing: if $x < y$, then $b^x < b^y$;
- if $0 < b < 1$, then b^x is decreasing: if $x < y$, then $b^x > b^y$.

Moreover, for $b > 1$,

$$\lim_{x \to \infty} b^x = \infty, \qquad \lim_{x \to -\infty} b^x = 0.$$

Examples from the real world

Population

Let $P(x)$ be the population at the year x, and assume

$$P(2026) = 8.3 \text{ (billion)},$$

with a growth rate of 0.84% (assume it is constant). Then

$$P(2027) = 8.3 \cdot 1.0084, \qquad P(2028) = 8.3 \cdot 1.0084^2,$$

and in general

$$P(x) = 8.3 \cdot 1.0084^{x-2026}.$$

Radioactivity

Caesium-137 is radioactive. Each atom of Cs-137 decays into Ba-137 by emitting an electron, with some probability. If there are X atoms, after 30 years half of them have decayed. So

$$f(t) = X \cdot \left(\frac{1}{2}\right)^{\frac{t}{30}}$$

is the number of Cs-137 atoms after t years.

Interest on an investment

If you have an amount of money P in your bank account for investment, with interest rate r per year, then next year you have

$$B = P(1 + r).$$

After t years,

$$B(t) = P(1 + r)^t.$$

Instead of annual interest, let us see what happens if we compound the interest monthly,

$$P\left(1+\frac{r}{12}\right)^{12t},$$

or daily,

$$P\left(1+\frac{r}{365}\right)^{365t}.$$

Napier's number

What is the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}?$$

n	10	100	1000
$\left(1+\frac{1}{n}\right)^{n}$	2.5937	2.7048	2.7169

The limit is

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e = 2.71828\ldots,$$

called Napier's number.

Exponential vs polynomial

Recall that

$$(a+b)^2 = a^2+2ab+b^2, \qquad (a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

Expanding $(1+1)^n$ in the same way,

$$2^n = (1+1)^n = 1+n+\frac{n(n-1)}{2}+\cdots,$$

so, for $n \geq 2$,

$$\frac{n}{2^n} = \frac{n}{1+n+\frac{n(n-1)}{2}+\cdots} < \frac{n}{\frac{n(n-1)}{2}} = \frac{2}{n-1},$$

and hence

$$\lim_{n\to\infty}\frac{n}{2^n} = 0.$$

In general, for $m > 0$ and $a > 1$,

$$\lim_{x\to\infty}\frac{x^m}{a^x} = 0.$$

Exercises

1. Simplify $27^{\frac{5}{3}}$ and $(3^2)^{\frac{5}{2}}$.
2. Find x such that $4^{2x-1} = 16$.
3. Find x such that $3^x \cdot 2^{2x} = 144$.