CHAPTER 4

PROBABILITY

Opening Example

Do you worry about your weight? According to a Gallup poll, 45% of American adults worry all or some of the time about their weight. The poll showed that more women than men worry about their weight all or some of the time. In this poll, 55% of adult women and 35% of adult men said that they worry all or some of the time about their weight. (See Case Study 4–1.)

4.1 Experiment, Outcomes, and Sample Space

Definition

An ***experiment*** is a process that, when performed, results in one and only one of many observations. These observations are called the ***outcomes*** of the experiment. The collection of all outcomes for an experiment is called a ***sample space***.

Table 4.1 Examples of Experiments, Outcomes, and Sample Spaces

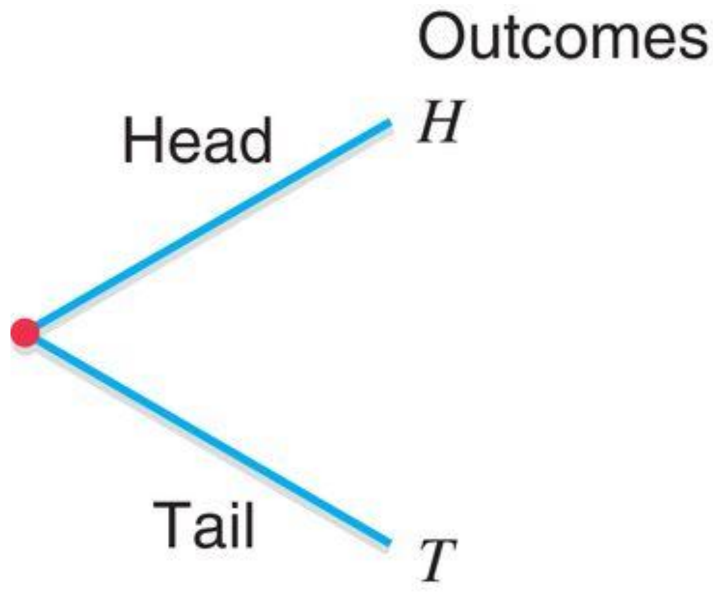
Experiment	Outcomes	Sample Space
Toss a coin once	Head, Tail	$S = \{\text{Head, Tail}\}$
Roll a die once	1, 2, 3, 4, 5, 6	$S = \{1, 2, 3, 4, 5, 6\}$
Toss a coin twice	HH, HT, TH, TT	$S = \{HH, HT, TH, TT\}$
Play lottery	Win, Lose	$S = \{\text{Win, Lose}\}$
Take a test	Pass, Fail	$S = \{\text{Pass, Fail}\}$
Select a worker	Male, Female	$S = \{\text{Male, Female}\}$

Example 4-1

Draw the tree diagram for the experiment of tossing a coin once.

Example 4-1: Solution

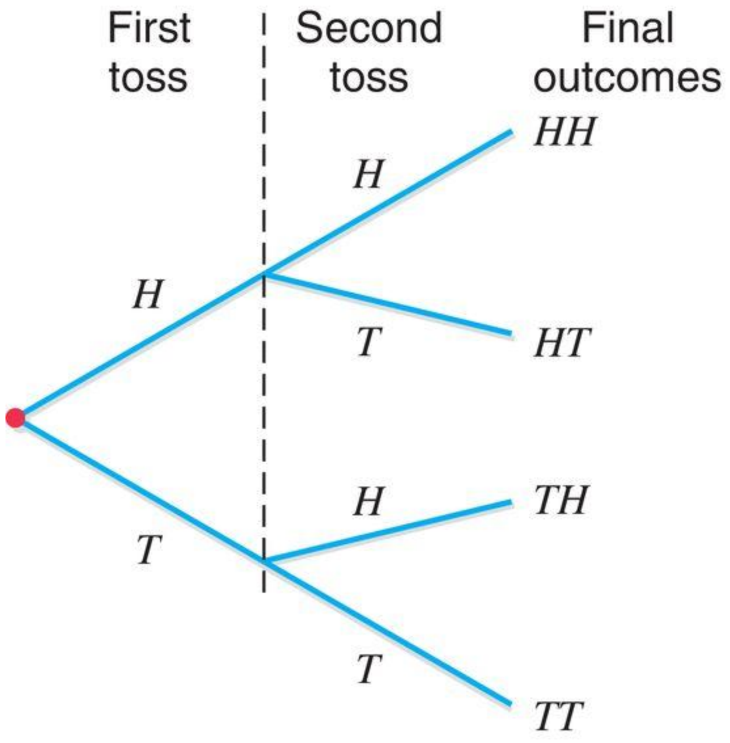
Figure 4.1 Tree Diagram for One Toss of a Coin



Example 4-2

Draw the tree diagram for the experiment of tossing a coin twice.

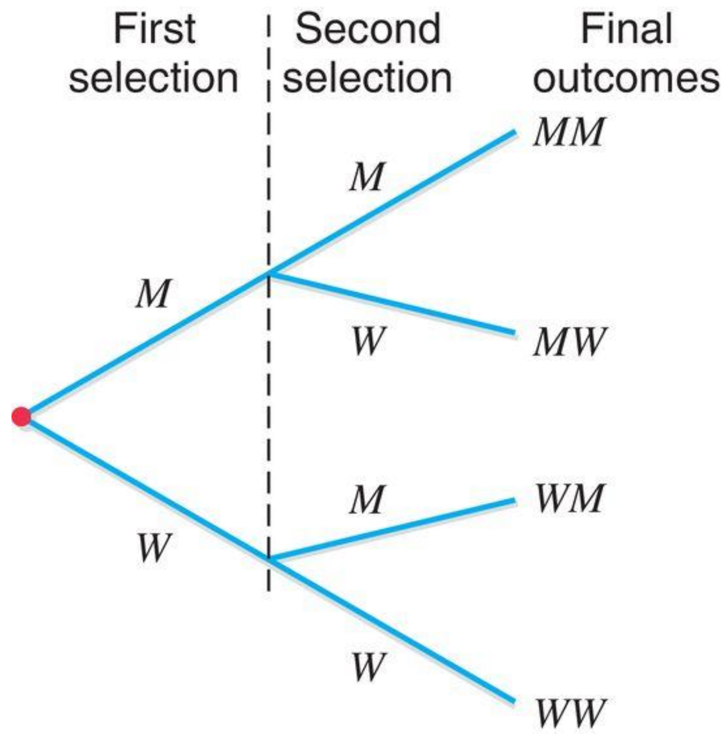
Example 4-2: Solution
Figure 4.2 Tree Diagram for Two Tosses of a Coin



Example 4-3

Suppose we randomly select two workers from a company and observe whether the worker selected each time is a man or a woman. Write all the outcomes for this experiment. Draw the tree diagram for this experiment.

Example 4.3: Solution
Figure 4.3 Tree Diagram for Selecting Two Workers



Event, Simple and Compound Events

Definition

An ***event*** is a collection of one or more of the outcomes of an experiment.

Event, Simple and Compound Events

Definition

An event that includes one and only one of the (final) outcomes for an experiment is called a ***simple event*** and is denoted by E_i.

Example 4-4

Reconsider Example 4-3 on selecting two workers from a company and observing whether the worker selected each time is a man or a woman. Each of the final four outcomes (*MM, MW, WM*, and *WW*) for this experiment is a simple event. These four events can be denoted by E_1, E_2, E_3, and E_4, respectively. Thus,

$$\boldsymbol{E_1} = \{MM\},\ \ \boldsymbol{E_2} = \{MW\},\ \ \boldsymbol{E_3} = \{WM\},\ \text{ and }\ \boldsymbol{E_4} = \{WW\}$$

Event, Simple and Compound Events

Definition

A ***compound event*** is a collection of more than one outcome for an experiment.

Example 4-5

Reconsider Example 4-3 on selecting two workers from a company and observing whether the worker selected each time is a man or a woman. Let *A* be the event that at most one man is selected. Is event *A* a simple or a compound event?

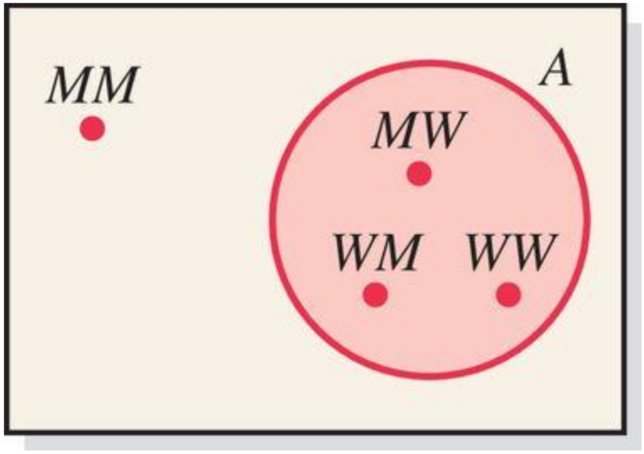
Example 4-5: Solution

Here *at most one man* means one or no man is selected. Thus, event *A* will occur if either no man or one man is selected. Hence, the event *A* is given by

$$A = \{MW, WM, WW\}$$

Because event *A* contains more than one outcome, it is a compound event. The diagram in Figure 4.4 gives a graphic presentation of compound event *A*.

Figure 4.4 Diagram Showing Event A



Example 4-6

In a group of college students, some like ice tea and others do not. There is no student in this group who is indifferent or has no opinion. Two students are randomly selected from this group.

(a) How many outcomes are possible? List all the possible outcomes.

(b) Consider the following events. List all the outcomes included in each of these events. Mention whether each of these events is a simple or a compound event.

(i) Both students like ice tea.

(ii) At most one student likes ice tea.

(iii) At least one student likes ice tea.

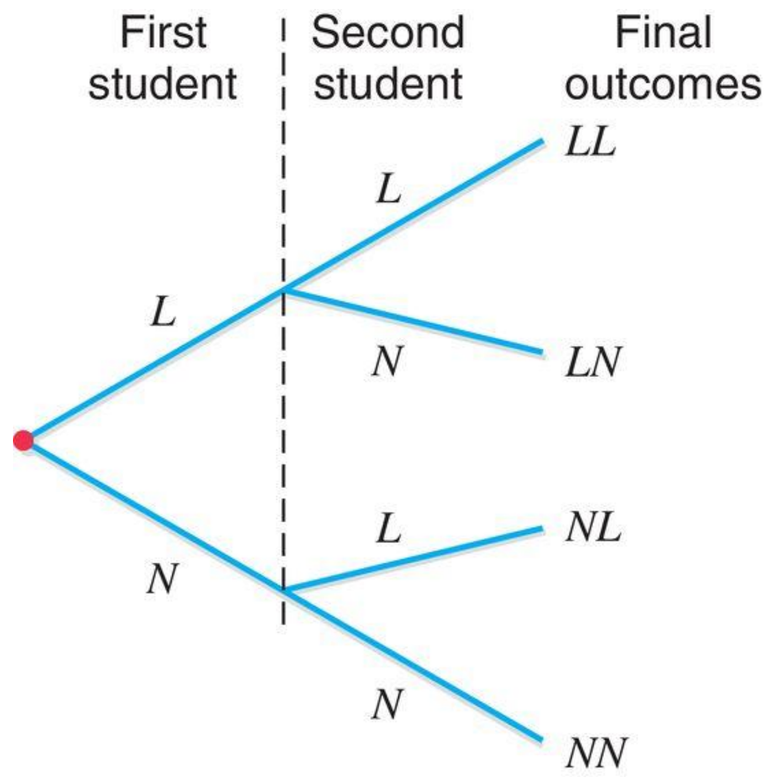
(iv) Neither student likes ice tea.

Example 4-6: Solution

Let

- *L* The event that a student likes ice tea
- *N* The event that a student does not like ice tea
- *LL* Both students like ice tea
- *LN* The first student likes ice tea but the second student does not
- *NL* The first student does not like ice tea but the second student does
- *LN* Both students does not like ice tea

Figure 4.5 Tree Diagram



Example 4-6: Solution

(a) This experiment has four outcomes, which are listed below and shown in Figure 4.5.

LL, LN, NL, NN

Example 4-6: Solution

(b)

(i) The event *both students like ice tea* will occur if *LL* happens. Thus,

Both students like ice tea = {*LL*}

Since this event includes only one of the four outcomes, it is a **simple** event.

Example 4-6: Solution

(b)

(ii) The event *at most one student likes ice tea* will occur if one or none of the two students likes ice tea, which will include the events *LN, NL,* and *NN*. Thus,

At most one student likes ice tea = {*LN*, *NL*, *NN*}

Since this event includes three outcomes, it is a **compound** event.

Example 4-6: Solution

(b)

(iii) The event *at least one student likes ice tea* will occur if one or two of the two students like ice tea, which will include the events *LN, NL,* and *LL*. Thus,

At least one student likes ice tea = {*L N*, *NL*, *LL*}

Since this event includes three outcomes, it is a **compound** event.

Example 4-6: Solution

(b)

(iv) The event *neither student likes ice tea* will occur if neither of the two students likes ice tea, which will include the event *NN*. Thus,

Neither student likes ice tea = {*NN*}

Since this event includes one outcome, it is a **simple** event.

4.2 Calculating Probability

Definition

Probability is a numerical measure of the likelihood that a specific event will occur.

Two Properties of Probability

- The probability of an event always lies in the range 0 to 1.

First Property of Probability

$$0 \leq P(E_i) \leq 1$$

$$0 \leq P(A) \leq 1$$

- The sum of the probabilities of all simple events (or final outcomes) for an experiment, denoted by $\boldsymbol{\Sigma P(E_i)}$, is always 1.

Second Property of Probability For an experiment,

$$\Sigma P(E_i) = P(E_1) + P(E_2) + P(E_3) + \cdots = 1$$

Three Conceptual Approaches to Probability

Classical Probability

Definition

Two or more outcomes that have the same probability of occurrence are said to be ***equally likely outcomes***.

Classical Probability

Classical Probability Rule to Find Probability

$$P(E_i) = \frac{1}{\text{Total number of outcomes for the experiment}}$$

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}}$$

Example 4-7

Find the probability of obtaining a head and the probability of obtaining a tail for one toss of a coin.

Example 4-7: Solution

$$P(\text{head}) = \frac{1}{\text{Total number of outcomes}} = \frac{1}{2} = .50$$

Similarly,

$$P(\text{tail}) = \frac{1}{2} = .50$$

Example 4-8

Find the probability of obtaining an even number in one roll of a die.

Example 4-8: Solution

A = an even number is obtained = {2, 4, 6}.

If any one of these three numbers is obtained, event A is said to occur. Hence,

$$P(\text{head}) = \frac{\text{Number of outcomes included in } A}{\text{Total number of outcomes}} = \frac{3}{6} = .50$$

Example 4-9

Jim and Kim have been looking for a house to buy in New Jersey. They like five of the homes they have looked at recently and two of those are in West Orange. They cannot decide which of the five homes they should pick to make an offer. They put five balls (of the same size) marked 1 through 5 (each number representing a home) in a box and asked their daughter to select one of these balls. Assuming their daughter's selection is random, what is the probability that the selected home is in West Orange?

Example 4-9: Solution

With random selection, each home has the same probability of being selected and the five outcomes (one for each home) are equally likely. Two of the five homes are in West Orange. Hence,

$$\boldsymbol{P(selected\ home\ is\ in\ West\ Orange) = \frac{2}{5} = .40}$$

Three Conceptual Approaches to Probability

Relative Frequency Concept of Probability

Using Relative Frequency as an Approximation of Probability

If an experiment is repeated n times and an event *A* is observed f times where f is the frequency, then, according to the relative frequency concept of probability:

$$P(A) = \frac{f}{n} = \frac{\text{Frequency of A}}{\text{Sample Size}}$$

Example 4-10

Ten of the 500 randomly selected cars manufactured at a certain auto factory are found to be lemons. Assuming that the lemons are manufactured randomly, what is the probability that the next car manufactured at this auto factory is a lemon?

Example 4-10: Solution

Let n denote the total number of cars in the sample and f the number of lemons in n. Then,

$$n = 500 \quad \text{and} \quad f = 10$$

Using the relative frequency concept of probability, we obtain

$$P(\text{next car is a lemon}) = \frac{f}{n} = \frac{10}{500} = .02$$

Table 4.2 Frequency and Relative Frequency Distributions for the Sample of Cars

Car	f	Relative Frequency
Good	490	$490/500 = .98$
Lemon	10	$10/500 = .02$
	$n = 500$	Sum $= 1.00$

Law of Large Numbers

Definition

Law of Large Numbers

If an experiment is repeated again and again, the probability of an event obtained from the relative frequency approaches the actual (or theoretical) probability.

Table 4.3 Simulating the Tosses of a Coin

Table 4.3 **Simulating the Tosses of a Coin**

Number of Tosses	Number of Heads	Number of Tails	$P(H)$	$P(T)$
3	0	3	.00	1.00
8	6	2	.75	.25
25	9	16	.36	.64
100	61	39	.61	.39
1000	522	478	.522	.478
10,000	4962	5038	.4962	.5038
1,000,000	500,313	499,687	.5003	.4997

If the experiment is repeated only a few times, the probabilities obtained may not be close to the actual probabilities. As the number of repetitions increases, the probabilities of outcomes obtained become very close to the actual probabilities.

Example 4-11

Allison wants to determine the probability that a randomly selected family from New York State owns a home. How can she determine this probability?

Example 4-11: Solution

There are two outcomes for a randomly selected family from New York State: “This family owns a home” and “This family does not own a home.” These two events are not equally likely. Hence, the classical probability rule cannot be applied. However, we can repeat this experiment again and again. Hence, we will use the relative frequency approach to probability.

Three Conceptual Approaches to Probability

Subjective Probability

Definition

Subjective probability is the probability assigned to an event based on subjective judgment, experience, information, and belief. There are no definite rules to assign such probabilities.

4.3 Marginal Probability, Conditional Probability, and Related Probability Concepts

Suppose all 100 employees of a company were asked whether they are in favor of or against paying high salaries to CEOs of U.S. companies. Table 4.4 gives a two way classification of the responses of these 100 employees. Assume that every employee responds either *in favor* or *against*.

Table 4.4 Two-Way Classification of Employee Responses

	In Favor	Against
Male	15	45
Female	4	36

Table 4.5 Two-Way Classification of Employee Responses with Totals

	In Favor	Against	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

Marginal Probability, Conditional Probability, and Related Probability Concepts

Definition

Marginal probability is the probability of a single event without consideration of any other event.

Table 4.6 Listing the Marginal Probabilities

Table 4.6 Listing the Marginal Probabilities

	In Favor (A)	Against (B)	Total
Male (M)	15	45	60
Female (F)	4	36	40
Total	19	81	100

$P(M) = 60/100 = .60$

$P(F) = 40/100 = .40$

$P(A) = 19/100 = .19$

$P(B) = 81/100 = .81$

Marginal Probability, Conditional Probability, and Related Probability Concepts

Read as "given"

$P(\text{in favor} \mid \text{male})$

The event whose probability is to be determined

This event has already occurred

Marginal Probability, Conditional Probability, and Related Probability Concepts

Definition

Conditional probability is the probability that an event will occur given that another has already occurred. If A and B are two events, then the conditional probability of A given B is written as

$$\boldsymbol{P(A \mid B)}$$

and read as "the probability of A given that B has already occurred."

Example 4-12

Compute the conditional probability P (in favor | male) for the data on 100 employees given in Table 4.5.

Example 4-12: Solution

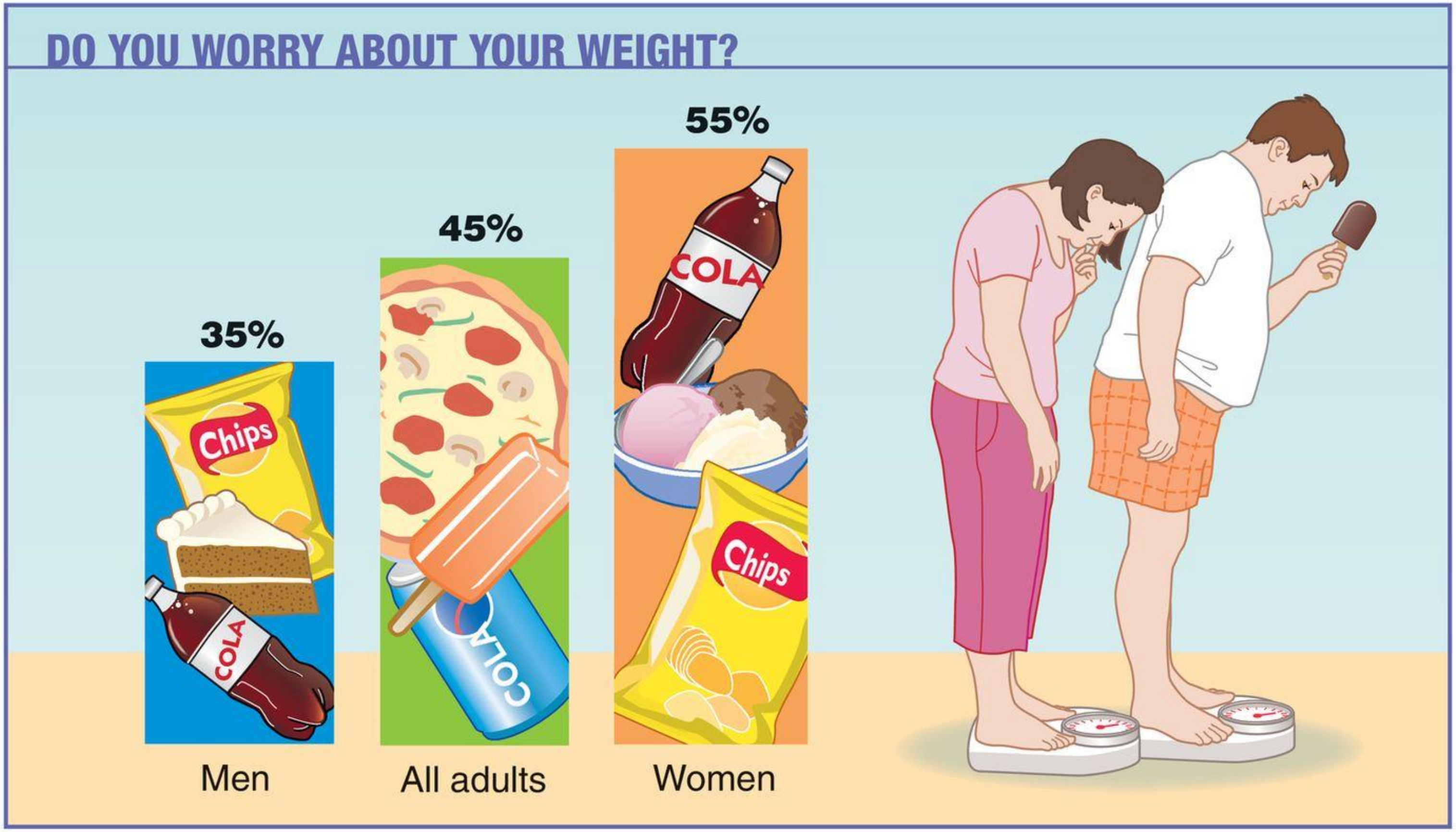
	In Favor	Against	Total
Male	15	45	60

Males who are in favor

Total number of males

$$P(\text{in favor} \mid \text{male}) = \frac{\text{Number of males who are in favor}}{\text{Total number of males}} = \frac{15}{60} = .25$$

Case Study 4-1 Do You Worry about Your Weight?



Data source: Gallup poll of 1013 adults aged 18 and older conducted July 7–10, 2014

Example 4-13

For the data of Table 4.5, calculate the conditional probability that a randomly selected employee is a female given that this employee is in favor of paying high salaries to CEOs.

Example 4-13: Solution

In Favor	
15	
4	← Females who are in favor
19	← Total number of employees who are in favor

$$P(\text{female} \mid \text{in favor}) = \frac{\text{Number of females who are in favor}}{\text{Total number of employees who are in favor}}$$

$$= \frac{4}{19} = .2105$$

Marginal Probability, Conditional Probability, and Related Probability Concepts

Definition

Events that cannot occur together are said to be ***mutually exclusive events***.

Example 4-14

Consider the following events for one roll of a die:

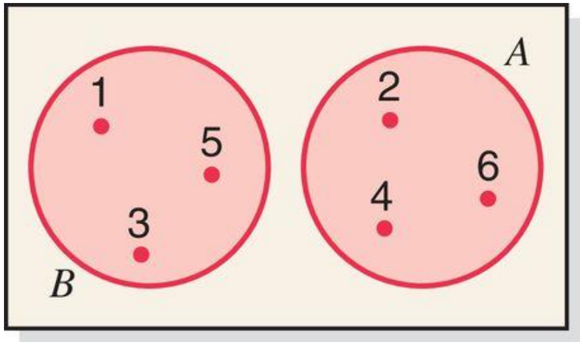
A= an even number is observed= {2, 4, 6}

B= an odd number is observed= {1, 3, 5}

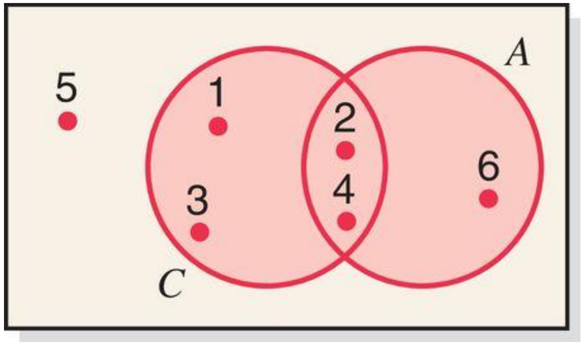
C= a number less than 5 is observed= {1, 2, 3, 4}

Are events A and B mutually exclusive? Are events A and C mutually exclusive?

Example 4-14: Solution



Mutually exclusive events *A* and *B*



Mutually nonexclusive events *A* and *C*

Example 4-14: Solution

We can observe from the definitions of events *A* and *C* and from Figure 4.7 that events *A* and *C* have two common outcomes: 2-spot and 4-spot. Thus, if we roll a die and obtain either a 2-spot or a 4-spot, then *A* and *C* happen at the same time. Hence, events *A* and *C* are not mutually exclusive.

Example 4-15

Consider the following two events for a randomly selected adult:

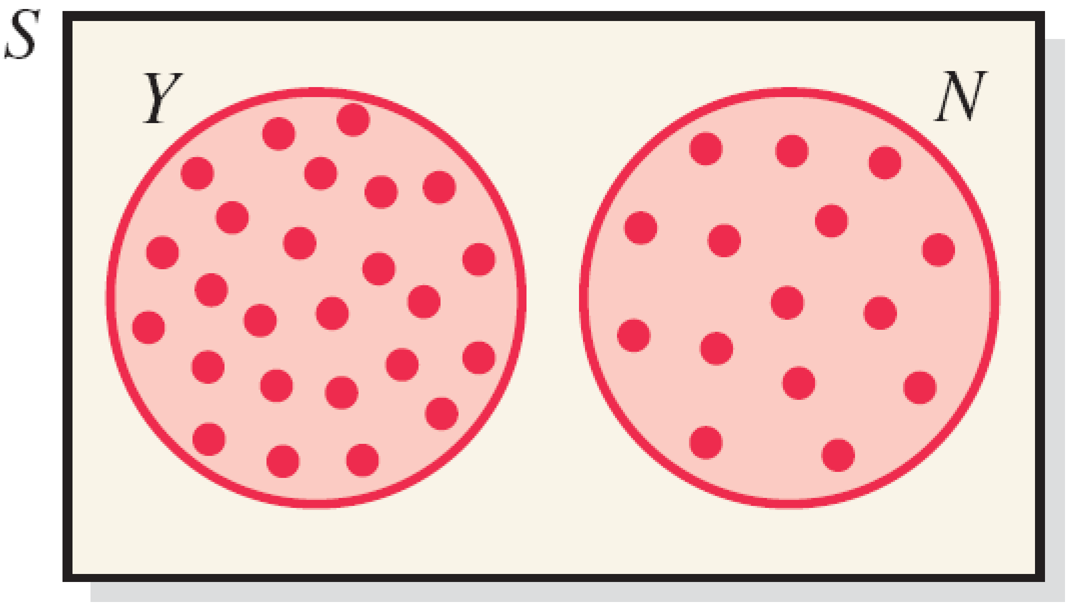
Y = this adult has shopped on the Internet at least once

N = this adult has never shopped on the Internet

Are events Y and N mutually exclusive?

Example 4-15: Solution

As we can observe from the definitions of events Y and N and from Figure 4.8 events Y and N have no common outcome. Hence, these two events are mutually exclusive.



Marginal Probability, Conditional Probability, and Related Probability Concepts

Definition

Two events are said to be ***independent*** if the occurrence of one does not affect the probability of the occurrence of the other. In other words, *A* and *B* are ***independent events*** if

$$\text{either} \quad P(A \mid B) = P(A) \quad \text{or} \quad P(B \mid A) = P(B)$$

Example 4-16

Refer to the information on 100 employees given in Table 4.5 in Section 4.3.1. Are events “in favor” and “female” independent?

Example 4-16: Solution

To check whether events *in favor* and *female* are independent, we find the (marginal) probability of *in favor*, and then the conditional probability of *in favor* given that *female* has already happened. If these two probabilities are equal then these two events are independent, otherwise they are dependent.

From Table 4.5, we compute the following two probabilities:

$$P\text{ (in favor)} = 19/100 = .19 \text{ and}$$

$$P\text{ (in favor | female)} = 4/40 = .1$$

Because these two probabilities are not equal, the two events are dependent.

Example 4-17

In a survey, 500 randomly selected adults who drink coffee were asked whether they usually drink coffee with or without sugar. Of these 500 adults, 240 are men, and 175 drink coffee without sugar. Of the 240 men, 84 drink coffee without sugar. Are the events drinking coffee *without sugar* and *man* independent?

Example 4-17: Solution

To check if the events drinking coffee *without sugar* and *man* are independent, first we find the probability that a randomly selected adult from these 500 adults drinks coffee *without sugar* and then find the conditional probability that this selected adult drinks coffee *without sugar* given that this adult is a *man*.

From the given information:

P (drinks coffee without sugar) = 175/500 = .35 and

P (drinks coffee without sugar | man) = 84/240 = .35

Because these two probabilities are equal, the two events drinking coffee *without sugar* and *man* are independent.

Example 4-17: Solution
Table 4.7 Gender and Drinking Coffee with or without Sugar

Table 4.7 Gender and Drinking Coffee with or without Sugar

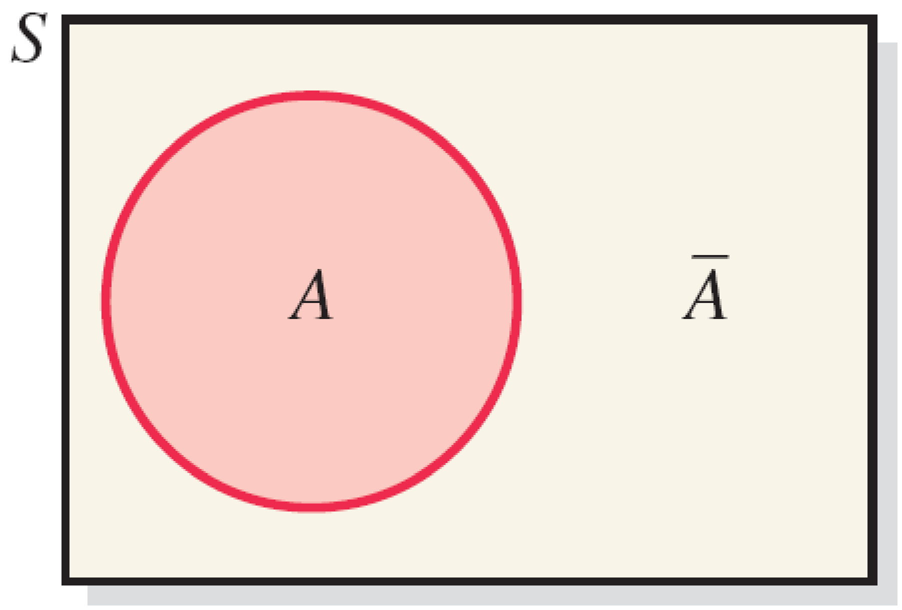
	With Sugar	Without Sugar	Total
Men	156	$\boxed{84}$	$\boxed{240}$
Women	169	91	260
Total	325	$\boxed{175}$	$\boxed{500}$

Marginal Probability, Conditional Probability, and Related Probability Concepts

Definition

The ***complement of event A***, denoted by $\bar{A}$ and read as "*A* bar" or "*A* complement," is the event that includes all the outcomes for an experiment that are not in *A*.

Figure 4.9 Two Complementary Events



Example 4-18

In a group of 2000 taxpayers, 400 have been audited by the IRS at least once. If one taxpayer is randomly selected from this group, what are the two complementary events for this experiment, and what are their probabilities?

Example 4-18: Solution

The two complementary events for this experiment are

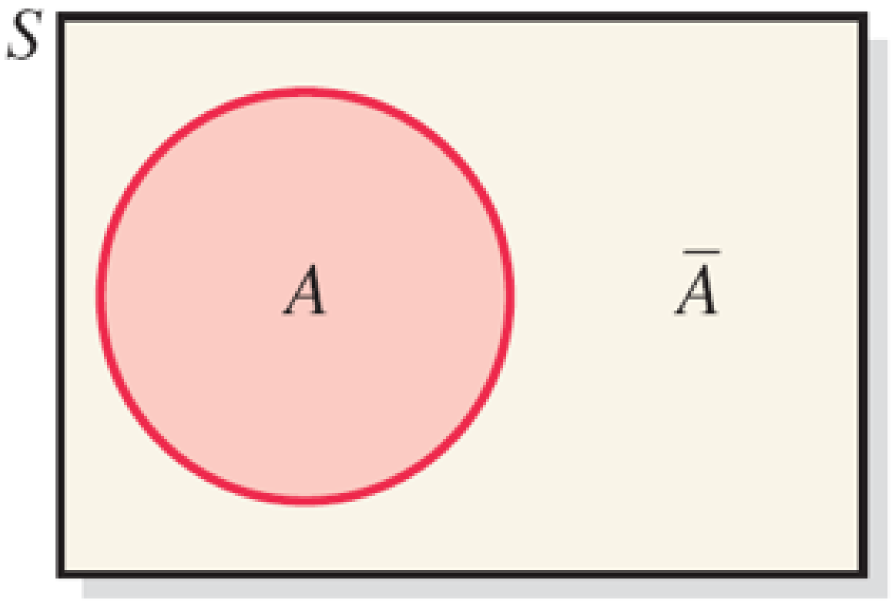
- A = the selected taxpayer has been audited by the IRS at least once
- $\bar{A}$ = the selected taxpayer has never been audited by the IRS

The probabilities of the complementary events are

$$P(A) = 400/2000 = .20 \quad \text{and}$$

$$P(\bar{A}) = 1600/2000 = .80$$

Figure 4.10 Complementary Events A and $\bar{A}$



Example 4-19

Refer to the information given in Table 4.7. Of the 500 adults, 325 drink coffee with sugar. Sup- pose one adult is randomly selected from these 500 adults, and let *A* be the event that this adult drinks coffee with sugar. What is the complement of event *A*? What are the probabilities of the two events?

Example 4-19: Solution

The two complementary events for this experiment are

A = the selected adult drinks coffee with sugar

$\bar{A}$ = the selected adult drinks coffee without sugar

The probabilities of the complementary events are

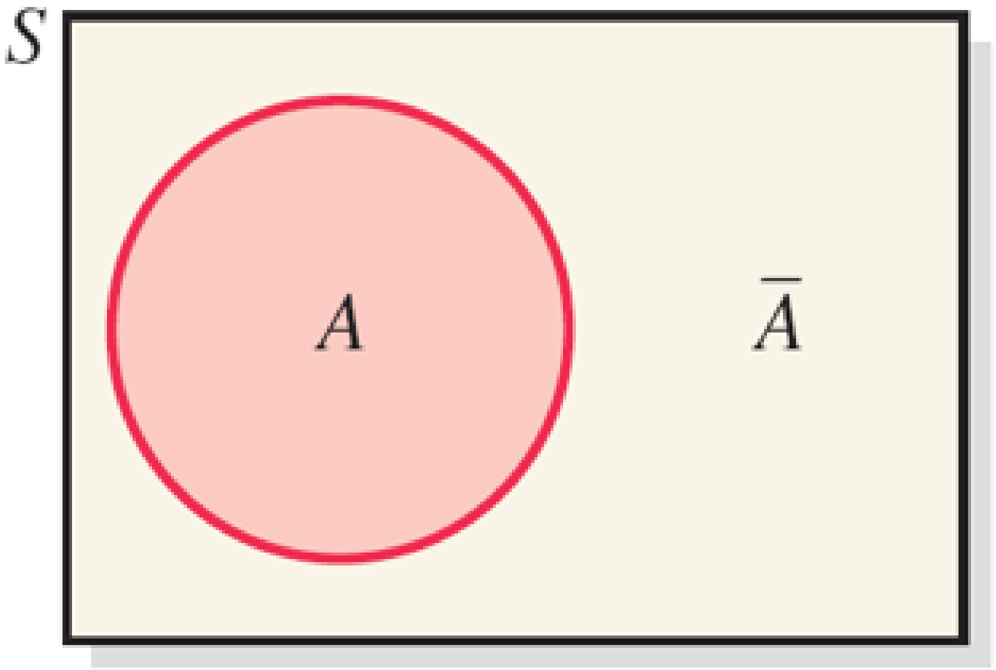
$P(A) = 325/500 = .65$ and

$P(\bar{A}) = 175/500 = .35$

Also we can observe, the sum of these two probabilities is 1, which should be the case with two complementary events. Also, once we find $P(A)$, we can find $P(\bar{\mathbf{A}})$ as:

$$P(\bar{\mathbf{A}}) = 1 - P(\mathrm{A}) = 1 - .65 = .35$$

Figure 4.11 Complementary Events Drinking Coffee with or without Sugar



4.4. Intersection of Events and the Multiplication Rule

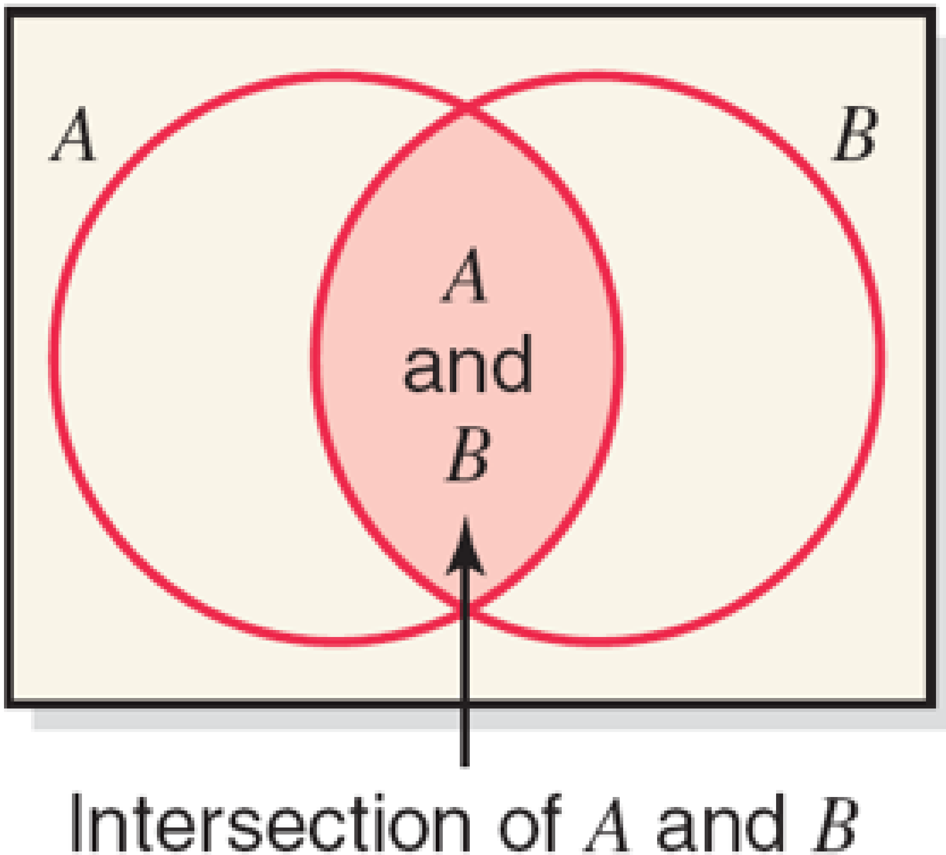
Intersection of Events

Definition

Let A and B be two events defined in a sample space. The ***intersection*** of A and B represents the collection of all outcomes that are common to both A and B and is denoted by

$$(A \text{ and } B)$$

Figure 4.12 Intersection of Events *A* and *B*



Intersection of Events and the Multiplication Rule

Multiplication Rule

Definition

The probability of the intersection of two events is called their ***joint probability***. It is written as

$$P(A \text{ and } B)$$

Intersection of Events and the Multiplication Rule

Multiplication Rule to Calculate the Probability of Independent Events

The probability of the intersection of two independent events A and B is

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example 4-20

An office building has two fire detectors. The probability is .02 that any fire detector of this type will fail to go off during a fire. Find the probability that both of these fire detectors will fail to go off in case of a fire. Assume that these two fire detectors are independent of each other.

Example 4-20: Solution

We define the following two events:

A = the first fire detector fails to go off during a fire
B = the second fire detector fails to go off during a fire

Then, the joint probability of A and B is

$$P(A \text{ and } B) = P(A) \times P(B) = (.02) \times (.02) = .0004$$

Example 4-21

The probability that a college graduate will find a full-time job within three months after graduation from college is .27. Three college students, who will be graduating soon, are randomly selected. What is the probability that all three of them will find full-time jobs within three months after graduation from college?

Example 4-21: Solution

Let A, B, and C denote the events the first, second, and third student, respectively, finds a full-time job within three months after graduation from college. Hence,

$$P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C)$$
$$= (.27) \times (.27) \times (.27) = .0197$$

Intersection of Events and the Multiplication Rule

Multiplication Rule to Find Joint Probability of Two Dependent Events

The probability of the intersection of two dependent events *A* and *B* is

$$P(A \text{ and } B) = P(A) \times P(B \mid A) \quad \text{or} \quad P(B) \times P(A \mid B)$$

Example 4-22

Table 4.8 gives the classification of all employees of a company given by gender and college degree.

	College Graduate (*G*)	Not a College Graduate (*N*)	Total
Male (*M*)	7	20	27
Female (*F*)	4	9	13
Total	11	29	40

Example 4-22

If one of these employees is selected at random for membership on the employee-management committee, what is the probability that this employee is a female and a college graduate?

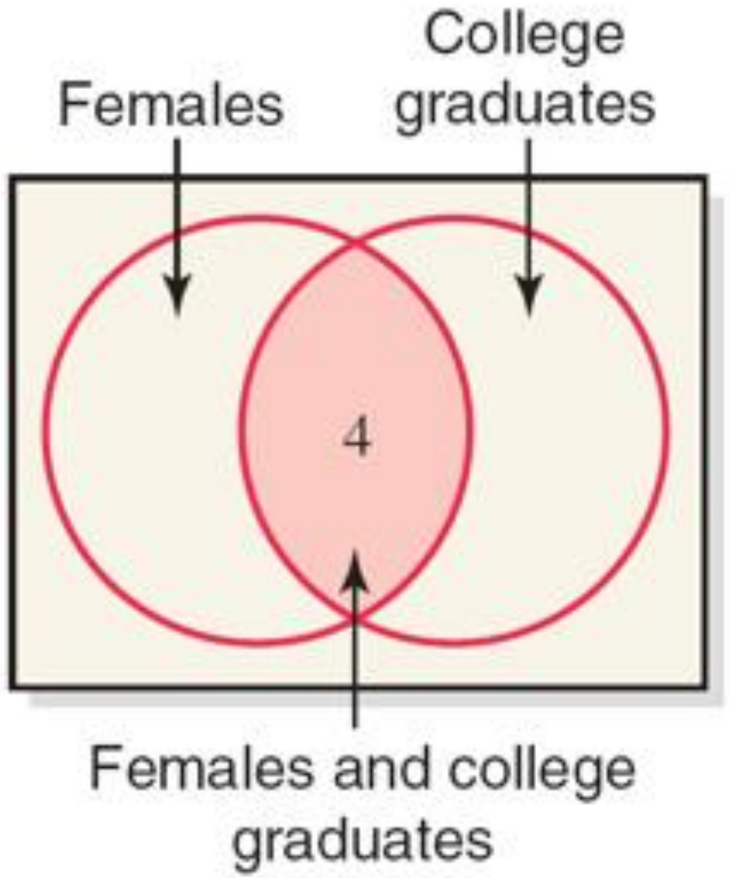
Example 4-22: Solution

We are to calculate the probability of the intersection of the events *female* and *college graduate.*

$P(\textit{female}) = 13/40$
$P(\textit{college graduate} \mid \textit{female}) = 4/13$

$$P(\textit{female} \text{ and } \textit{college graduate}) = P(\textit{female}) \times P(\textit{college graduate} \mid \textit{female}) = (13/40) \times (4/13) = .10$$

Figure 4.13 Intersection of Events *Female* and *College Graduate*



Example 4-23

In a group of 20 college students, 7 like ice tea and others do not. Two students are randomly selected from this group.

(a) Find the probability that both of the selected students like ice tea.

(b) Find the probability that the first student selected likes ice tea and the second does not.

Example 4-23: Solution

(a) From the given information:

P(*first student likes ice tea*) = 7/20 = .35

P(*second student likes ice tea* | *first student likes ice tea*)

= 6/19 = .3158

Thus, the joint probability of the two events is:

P(both students like ice tea)

= *P*(*first student likes ice tea*) x

P(*second student likes ice tea* | *first student likes ice tea*)

= (.35) x (.3158) = **.1105**

Example 4-23: Solution

(b) From the given information:

$P(1^{st}$ *student likes ice tea*$) = 7/20 =$ **.35**

$P(2^{nd}$ *student does not like ice tea* | 1^{st} *student likes ice tea*$)$

$= 13/19 =$ **.6842**

Thus, the joint probability of the two events is:

$P(1^{st}$ *student likes ice tea and* 2^{nd} *student does not like it*$)$

$= P(1^{st}$ *student likes ice tea*$)$ x

$P(2^{nd}$ *student does not like it* | 1^{st} *student likes ice tea*$)$

$= (.35) \times (.6842) =$ **.2395**

Calculating Conditional Probability

If *A* and *B* are two events, then,

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \quad \text{and} \quad P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

given that $P(A) \neq 0$ and $P(B) \neq 0$.

Example 4-24

The probability that a randomly selected student from a college is a senior is .20, and the joint probability that the student is a computer science major and a senior is .03. Find the conditional probability that a student selected at random is a computer science major given that the student is a senior.

Example 4-24: Solution

From the given information,

P(senior) = **.20** and

P(senior and *computer science major)* = **.03**

Hence,

P(*computer science major* | *senior*)

= *P(senior* and *computer science major)* / *P(senior)*

= .03 / .20 = **.15**

Mutually Exclusive Events

Joint Probability of Mutually Exclusive Events

The joint probability of two mutually exclusive events is always zero. If A and B are two mutually exclusive events, then

$$P(A \text{ and } B) = 0$$

Example 4-25

Consider the following two events for an application filed by a person to obtain a car loan:

A = event that the loan application is approved

R = event that the loan application is rejected

What is the joint probability of A and R?

Example 4-25: Solution

The two events A and R are mutually exclusive. Either the loan application will be approved or it will be rejected. Hence,

$$P(A \text{ and } R) = \mathbf{0}$$

4.5 Union of Events and the Addition Rule

Definition

Let A and B be two events defined in a sample space. The ***union of events*** A and B is the collection of all outcomes that belong to either A or B or to both A and B and is denoted by

$$(A \text{ or } B)$$

Example 4-26

All 440 employees of a company were asked what hot drink they like the most. Their responses are listed in Table 4.9.

Describe the union of events *female* and likes *regular coffee*.

Table 4.9 Hot Drinks Employees Like

Table 4.9 Hot Drinks Employees Like

	Regular Coffee	Regular Tea	Other	Total
Male	110	70	110	290
Female	60	50	40	150
Total	170	120	150	440

Example 4-26: Solution

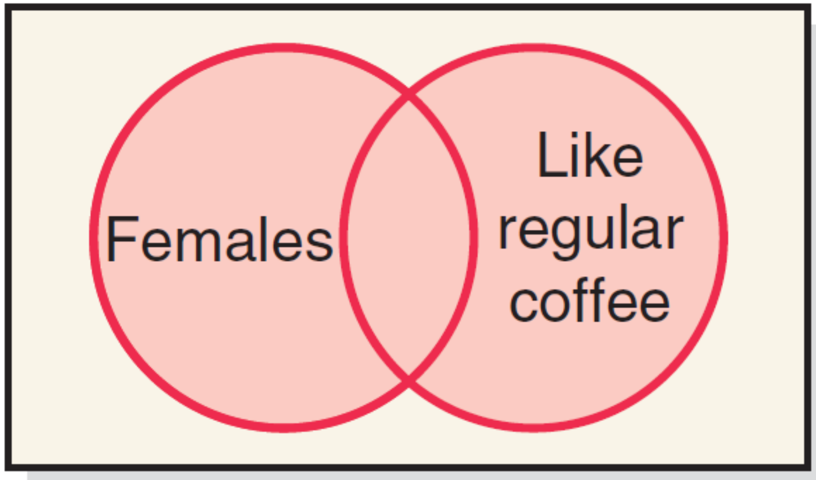
The union of events *female* and likes *regular coffee* includes those employees who are either female or like regular coffee or both. As we can observe from Table 4.9, there are a total of 150 females in these employees, and 170 of all employees like regular coffee.

Thus, the number of employees who are either female or like regular coffee or both is:

$$150 + 170 - 60 = \mathbf{260}$$

These 260 employees give the union of events *female* and likes *regular coffee*, which is shown in Figure 4.14.

Figure 4.14 Union of Events *Female* and *Likes Regular Coffee*



Area shaded in red gives the union of events *female* and likes *regular coffee*, and includes 260 employees who are either female or like regular coffee or both.

Addition Rule for Mutually Nonexclusive Events

Addition Rule

Addition Rule to Find the Probability of Union of Two Mutually Nonexclusive Events

The portability of the union of two mutually nonexclusive events A and B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 4-27

A university president has proposed that all students must take a course in ethics as a requirement for graduation. Three hundred faculty members and students from this university were asked about their opinion on this issue. Table 4.10 gives a two-way classification of the responses of these faculty members and students.

Find the probability that one person selected at random from these 300 persons is a faculty member or is in favor of this proposal.

Table 4.10 Two-Way Classification of Responses

	Favor	Oppose	Neutral	Total
Faculty	45	15	10	70
Student	90	110	30	230
Total	135	125	40	300

Example 4-27: Solution

From the information in the Table 4.10,

$P(faculty) = 70/300 = \mathbf{.2333}$

$P(in\ favor) = 135/300 = \mathbf{.4500}$

$P(faculty \text{ and } in\ favor) = 45/300 = \mathbf{.1500}$

Using the addition rule, we obtain

$P(faculty \text{ or } in\ favor)$

$= P(faculty) + P(in\ favor) - P(faculty \text{ and } in\ favor)$

$= .2333 + .4500 - .1500 = \mathbf{.5333}$

Example 4-28

In a group of 2500 persons, 1400 are female, 600 are vegetarian, and 400 are female and vegetarian. What is the probability that a randomly selected person from this group is a male or vegetarian?

Example 4-28: Solution

From the information in the Table 4.11,

$P(male) = 1100/2500 = \mathbf{.44}$

$P(vegetarian) = 600/2500 = \mathbf{.24}$

$P(male \text{ and } vegetarian) = 200/2500 = \mathbf{.08}$

Using the addition rule, we obtain

$$P(male \text{ or } vegetarian)$$
$$= P(male) + P(vegetarian) - P(male \text{ and } vegetarian)$$
$$= .44 + .24 - .08 = \mathbf{.60}$$

Addition Rule for Mutually Exclusive Events

Addition Rule to Find the Probability of the Union of Mutually Exclusive Events

The probability of the union of two mutually exclusive events A and B is

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 4-29

Refer to Example 4–27. A university president proposed that all students must take a course in ethics as a requirement for graduation. Three hundred faculty members and students from this university were asked about their opinion on this issue. The following table, reproduced from Table 4.10 in Example 4–27, gives a two-way classification of the responses of these faculty members and students.

What is the probability that a randomly selected person from these 300 faculty members and students is in favor of the proposal or is neutral?

Example 4-29: Solution

Table 4.10 Two-Way Classification of Responses

	Favor	Oppose	Neutral	Total
Faculty	45	15	10	70
Student	90	110	30	230
Total	135	125	40	300

Example 4-29: Solution

From the given information,

$P(\textit{in favor}) = 135/300 = \mathbf{.4500}$

$P(\textit{neutral}) = 40/300 = \mathbf{.1333}$

Hence,

$P(\textit{in favor} \text{ or } \textit{neutral}) = .4500 + .1333 = \mathbf{.5833}$

Example 4-30

Consider the experiment of rolling a die once. What is the probability that a number less than 3 or a number greater than 4 is obtained?

Example 4-30: Solution

Here, the event *a number less than 3 is obtained* happens if either 1 or 2 is rolled on the die, and the event *a number greater than 4 is obtained* happens if either 5 or 6 is rolled on the die. Thus, these two events are mutually exclusive as they do not have any common outcome and cannot happen together. Hence, their joint probability is zero.

P(*a number less than 3 is obtained*) = **2/6**

P(*a number greater than 4 is obtained*) = **2/6**

Example 4-30: Solution

The probability of the union of these two events is:

P(*a number less than 3 is obtained* or *a number greater than 4 is obtained*) = 2/6 + 2/6 = **.6667**

4.6 Counting Rule, Factorials, Combinations, and Permutations

Counting Rule to Find Total Outcomes

If an experiment consists of three steps and if the first step can result in m outcomes, the second step in n outcomes, and the third in k outcomes, then

$$\text{Total outcomes for the experiment} = m \cdot n \cdot k$$

Example 4-31

Consider three tosses of a coin. How many total outcomes this experiment has?

Example 4-31: Solution

This experiment of tossing a coin three times has three steps: the first toss, the second toss, and the third toss. Each step has two outcomes: a head and a tail. Thus,

Total outcomes for three tosses of a coin = 2 x 2 x 2 = **8**

The eight outcomes for this experiment are

HHH, *HHT*, *HTH*, *HTT*, *THH*, *THT*, *TTH*, and *TTT*

Example 4-32

A prospective car buyer can choose between a fixed and a variable interest rate and can also choose a payment period of 36 months, 48 months, or 60 months. How many total outcomes are possible?

Example 4-32: Solution

There are two outcomes (a fixed or a variable interest rate) for the first step and three outcomes (a payment period of 36 months, 48 months, or 60 months) for the second step. Hence,

Total outcomes = 2 x 3 = **6**

Example 4-33

A National Football League team will play 16 games during a regular season. Each game can result in one of three outcomes: a win, a loss, or a tie. How many total outcomes are possible?

Example 4-33: Solution

The total possible outcomes for 16 games are calculated as follows:

$$\text{Total outcomes} = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$
$$= 3^{16} = \mathbf{43{,}046{,}721}$$

One of the 43,046,721 possible outcomes is all 16 wins.

Counting Rule, Factorials, Combinations, and Permutations

Factorials

Definition

The symbol ***n!***, read as "*n* factorial," represents the product of all the integers from *n* to 1. In other words,

$$n! = n(n - 1)(n - 2)(n - 3) \cdot \cdot \cdot 3 \cdot 2 \cdot 1$$

By definition,

$$0! = 1$$

Example 4-34

Evaluate 7!

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{5040}$$

Example 4-35

Evaluate 10!

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= \mathbf{3{,}628{,}800}$$

Example 4-36

Evaluate (12-4)!

$$(12-4)! = 8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= \mathbf{40{,}320}$$

Example 4-37

Evaluate (5-5)!

$(5-5)! = 0! = \mathbf{1}$

Note that 0! is always equal to 1.

Combinations

Definition

Combinations give the number of ways x elements can be selected from n elements. The notation used to denote the total number of combinations is

$$_{n}C_{x}$$

which is read as "the number of combinations of n elements selected x at a time."

Combinations

n denotes the total number of elements

$${}_nC_x = \text{the number of combinations of } n \text{ elements selected } x \text{ at a time}$$

x denotes the number of elements selected per selection

Combinations

Number of Combinations

The ***number of combinations*** for selecting x from n distinct elements is given by the formula

$$_{n}C_{x} = \frac{n!}{x!(n-x)!}$$

where ***n!***, ***x!***, and ***(n-x)!*** are read as "*n factorial,*" "*x factorial,*" "*n minus x factorial*," respectively.

Example 4-38

An ice cream parlor has six flavors of ice cream. Kristen wants to buy two flavors of ice cream. If she randomly selects two flavors out of six, how many combinations are there?

Example 4-38: Solution

n = total number of ice cream flavors = 6

x = number of ice cream flavors to be selected = 2

$$ {}_6C_2 = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot4\cdot3\cdot2\cdot1} = 15 $$

Thus, there are **15** ways for Kristen to select two ice cream flavors out of six.

Example 4-39

Three members of a committee will be randomly selected from five people. How many different combinations are possible?

Example 4-39: Solution

n = 5 and x = 3

$$_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \frac{120}{6 \cdot 2} = 10$$

Example 4-40

Marv & Sons advertised to hire a financial analyst. The company has received applications from 10 candidates who seem to be equally qualified. The company manager has decided to call only 3 of these candidates for an interview. If she randomly selects 3 candidates from the 10, how many total selections are possible?

Example 4-40: Solution

n = 10 and *x* = 3

$$ {}_{10}C_3 = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{3,628,800}{(6)(5040)} = 120 $$

Thus, the company manager can select 3 applicants from 10 in **120** ways.

Case Study 4-2 Probability of Winning a Mega Millions Lottery Jackpot

Number of white balls matched	Number of gold balls matched	Prize	Number of white balls matched	Number of gold balls matched	Prize
5	1	Jackpot	3	0	$5
5	0	$1,000,000	2	1	$5
4	1	$5,000	1	1	$2
4	0	$500	0	1	$1
3	1	$50			

Permutations Notation

Permutations give the total selections of x element from n (different) elements in such a way that the order of selections is important. The notation used to denote the permutations is

$$_{n}P_{x}$$

which is read as "*the number of permutations of selecting x elements from n elements.*" Permutations are also called ***arrangements***.

Permutations Formula

The following formula is used to find the number of permutations or arrangements of selecting x items out of n items. Note that here, the n items must all be different.

$$ {}_{n}P_{x} = \frac{n!}{(n-x)!} $$

Example 4-41

A club has 20 members. They are to select three office holders – president, secretary, and treasurer – for next year. They always select these office holders by drawing 3 names randomly from the names of all members. The first person selected becomes the president, the second is the secretary, and the third one takes over as treasurer. Thus, the order in which 3 names are selected from the 20 names is important. Find the total arrangements of 3 names from these 20.

Example 4-41: Solution

n = total members of the club = 20

x = number of names to be selected = 3

$$ {}_{n}P_{x} = \frac{n!}{(n-x)!} = \frac{20!}{(20-3)!} = \frac{20!}{17!} = 6840 $$

Thus, there are **6840** permutations or arrangements for selecting 3 names out of 20.