1^{st} Assignment Statistics: Solutions (partial)

Maura Mezzetti

Exercise 5

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as Beta distribution.

$$f(x|\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 \le x \le 1$$

Show that the following statistics

$$T = \frac{1}{n}\left(\sum_{i=1}^{n} log(1-X_i)\right)^3$$

is a sufficient statistics for parameter β.

$$\begin{aligned} f(x_1, x_2, \ldots, x_n|\beta) &= \prod \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x_i^{\alpha-1}(1-x_i)^{\beta-1} \\ L(\beta|x_1, x_2, \ldots, x_n) &= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n \prod x_i^{\alpha-1} \prod (1-x_i)^{\beta-1} \end{aligned}$$

$$\left(\prod(1-x_i)\right)^{\beta-1}$$

contains both parameter β and the data, $T = \prod(1-x_i)$ is a sufficient statistics and, as a consequence, also the statistics $T_1 = \sum log(1-x_i)$ and $T_1 = (\sum log(1-x_i))^3$ is a sufficient statistics

Exercise 6

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as

$$f(x;\theta) = \theta(1+x)^{-(\theta+1)} I_{(0,\infty)}(x) \qquad \theta > 1$$

1. Find the MME and the MLE of θ
2. Find the MLE of $1/\theta$

1.

$$\begin{aligned}
f(x;\theta) &= \theta(1+x)^{-(\theta+1)} \\
E(X) &= \int_0^\infty x f(x;\theta)dx \\
E(X) &= \int_0^\infty x\theta(1+x)^{-(\theta+1)}dx = \frac{1}{\theta-1}
\end{aligned}$$

$$\hat{\theta}_{MOM} = \frac{1}{\bar{x}} + 1$$

$$\begin{aligned}
f(x_1, x_2, \ldots, x_n;\theta) &= \prod_i \theta(1+x_i)^{-(\theta+1)} \\
L(\theta|x_1, x_2, \ldots, x_n) &= \theta^n \prod_i (1+x_i)^{-(\theta+1)} \\
logL(\theta|x_1, x_2, \ldots, x_n) &= nlog\,(\theta) - (\theta+1)\sum_i log(1+x_i) \\
\frac{\partial logL(\theta|x_1, x_2, \ldots, x_n)}{\partial\theta} &= \frac{n}{\theta} - \sum_i log(1+x_i) \\
\frac{\partial^2 logL(\theta|x_1, x_2, \ldots, x_n)}{\partial^2\theta} &= -\frac{n}{\theta^2}
\end{aligned}$$

$\frac{\partial logL(\theta|x_1, x_2, \ldots, x_n)}{\partial\theta} = 0$ if $\hat{\theta}_{MLE} = \frac{n}{\sum_i log(1+x_i)}$

2.

$$\left(\frac{1}{\theta}\right)_{MLE} = \left(\frac{1}{\hat{\theta}_{MLE}}\right) = \frac{\sum_i log(1+x_i)}{n}$$