Gaussian distribution

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Sufficient Statistics

$$f(y_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}\left(y_i - \mu\right)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}\left(y_i - \mu\right)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2}$$

σ^2 known, Sufficient Statistics for μ

$$f(y_1,\ldots,y_i,\ldots,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2}\, e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i}\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

σ^2 known, Sufficient Statistics for μ

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2}$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

$$T(y) = \sum_{i=1}^{n} y_i \ \ \textit{sufficient statistics for } \mu$$

Sufficient Statistics for σ^2

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

Sufficient Statistics for σ^2

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$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

Sufficient Statistics for σ^2

$$f(y_1,\ldots,y_i,\ldots,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-\mu\right)^2}$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2}\, e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i}\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu}$$

$$T(y) = \left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} y_i^2\right) \quad \textit{sufficient statistics for } \sigma^2$$

Suppose that we have two variables

1. *Y* – the dependent variable (response variable)
2. *X* – the independent variable (explanatory variable, factor)-

X may or may not be a random variable .

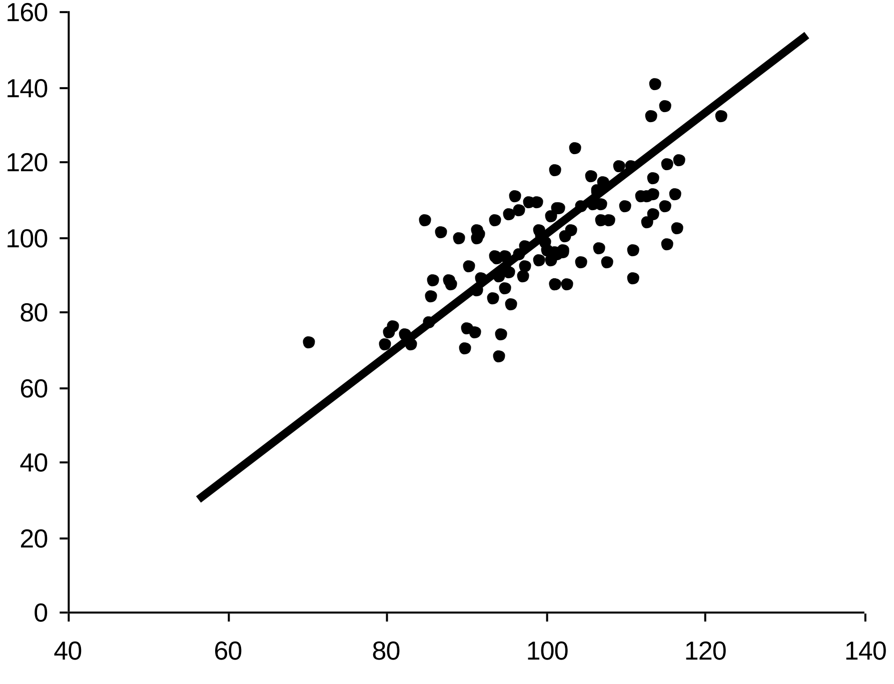
The dependent variable, *Y*, is assumed to be a random variable. The distribution of *Y* is dependent on *X*

Assume that we have collected data on two variables X and Y. Let

$$(x_1, y_1)\ (x_2, y_2)\ (x_3, y_3)\ \ldots\ (x_n, y_n)$$

denote the pairs of measurements on the on two variables X and Y for n cases in a sample (or population)

- When data is correlated it falls roughly about a straight line.



Regression Line

At a given value of x, the equation:

$$\hat{y} = \beta_0 + \beta_1 x$$

Predicts a single value of the response variable

But… we should not expect all subjects at that value of x to have the same value of y

Variability occurs in the y values!

Simple Linear Regression Model

- $y = \beta_0 + \beta_1 x + \varepsilon$
  - x: regressor variable
  - y: response variable
  - β_0: the intercept, unknown
  - β_1: the slope, unknown
  - ε : error with $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2$ (unknown)
- The errors are uncorrelated.

Linear Regression Model

- $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
  - x_i: regressor variable
  - y_i: response variable
  - β_0: the intercept, unknown
  - β_1: the slope, unknown
  - $\varepsilon_i \sim N(0, \sigma^2)$
- The errors are uncorrelated.

Sufficient Statistics

$$L(\sigma,\beta_0,\beta_1)=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-[\beta_0+\beta_1x_i]\right)^2}$$

$$=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i^2-2y_i[\beta_0+\beta_1x_i]+[\beta_0+\beta_1x_i]^2\right)}$$

$$=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}y_i^2-2\beta_0\sum_{i=1}^{n}y_i-2\beta_1\sum_{i=1}^{n}y_ix_i+n\beta_0{}^2+2\beta_0\beta_1\sum_{i=1}^{n}x_i+\beta_1{}^2\sum_{i=1}^{n}x_i^2\right)}$$

Sufficient Statistics

$$L(\sigma,\beta_0,\beta_1)=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-[\beta_0+\beta_1 x_i]\right)^2}$$

$$=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i^2-2y_i[\beta_0+\beta_1 x_i]+[\beta_0+\beta_1 x_i]^2\right)}$$

$$=\frac{1}{(2\pi)^{n/2}\sigma^n}e^{-\frac{n\beta_0{}^2}{2\sigma^2}}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}y_i^2-2\beta_0\sum_{i=1}^{n}y_i-2\beta_1\sum_{i=1}^{n}y_i x_i+n\beta_0{}^2+2\beta_0\beta_1\sum_{i=1}^{n}x_i+\beta_1{}^2\sum_{i=1}^{n}x_i^2\right)}$$

$$=h(x,y)c(\theta)\exp\left(\sum_{j=1}^{k}w_j(\theta)t_j(x,y)\right)$$

$$T_1=\sum_{i=1}^{n}y_i^2 \quad T_2=\sum_{i=1}^{n}y_i \quad T_3=\sum_{i=1}^{n}y_i x_i \quad T_4=\sum_{i=1}^{n}x_i \quad T_5=\sum_{i=1}^{n}x_i^2$$

Sufficient Statistics

$$L(\beta_0, \beta_1) =$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2\right)} e^{-\frac{1}{2\sigma^2}\left(-2\beta_0 \sum_{i=1}^{n} y_i - 2\beta_1 \sum_{i=1}^{n} y_i x_i + n\beta_0{}^2 + 2\beta_0\beta_1 \sum_{i=1}^{n} x_i + \beta_1{}^2 \sum_{i=1}^{n} x_i^2\right)}$$

$$T_1 = \sum_{i=1}^{n} y_i \quad T_2 = \sum_{i=1}^{n} y_i x_i \quad T_3 = \sum_{i=1}^{n} x_i \quad T_4 = \sum_{i=1}^{n} x_i^2$$