

Statistics: Exercise session 1

MSc Economics & MSc Finance and banking

TA: Tiziano Iannaccio
tiziano.iannaccio@uniroma1.it

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Exercise 1. The waiting time at the post office is distributed as an exponential distribution with mean 20 minutes.

- a) What is the probability that a client will wait more than 20 minutes? What would it be if the distribution were uniform?
- b) You will go to the post office next week every day from Monday to Friday, what is the probability that the maximum time you will wait is between 30 and 32 minutes?

Solution:

a) We have $X \sim \text{Exp}(\lambda)$ with $\mathbb{E}(X) = \frac{1}{\lambda} = 20$, which implies $\lambda = \frac{1}{20}$. The probability that a client waits more than 20 minutes is:

$$\begin{aligned}\mathbb{P}(X > 20) &= 1 - \mathbb{P}(X \leq 20) \\ &= 1 - F_X(20) \\ &= 1 - \left(1 - e^{-\frac{20}{20}}\right) \\ &= e^{-1} \simeq 0.3679\end{aligned}$$

A uniform distribution is symmetric around its mean (in this case $\mathbb{E}(X) = 20$). This implies that exactly half of its probability mass is allocated before (and after) the point $x = 20$. For this reason:

$$\mathbb{P}(X > 20) = \mathbb{P}(X \leq 20) = 0.5$$

To be more rigorous, assume $X \sim \text{Unif}(a, b)$ with $\mathbb{E}(X) = 20 = \frac{a+b}{2}$:

$$\begin{aligned}\mathbb{P}(X > 20) &= 1 - F_X(20) \\ &= 1 - \frac{20 - a}{b - a} \\ &= 1 - \frac{\frac{a+b}{2} - a}{b - a} \\ &= 1 - \frac{b - a}{2(b - a)} = 1 - \frac{1}{2} = \frac{1}{2}\end{aligned}$$

b) We have $X_i \stackrel{\text{iid}}{\sim} \text{Exp}(\lambda = \frac{1}{20})$ for $i = 1, \dots, 5$. In general, if we want to evaluate the probability of *the maximum* of a sequence of random variables ($\max\{X_1, \dots, X_n\}$) to be less than or equal to a given value $x \in \mathbb{R}$, we evaluate the joint probability that all random variables are less than or equal to x :

$$\begin{aligned} F_{\max\{X_1, \dots, X_n\}}(x; \lambda_1, \dots, \lambda_n) &= F_{X_1, \dots, X_n}(x, \dots, x; \lambda_1, \dots, \lambda_n) \\ &= \prod_{i=1}^n F_{X_i}(x; \lambda_i) \quad (\text{by independence}) \\ &= [F_X(x; \lambda)]^n \quad (\text{by identical distribution}) \end{aligned}$$

Note: we can replace x_i with x only because we are interested in a single value (the maximum is evaluated as less than or equal to a single value x). In general, you would need to replace $[F_X(x; \lambda)]^n$ with $\prod_{i=1}^n F_X(x_i, \lambda)$.

Using the cdf for $n = 5$:

$$\begin{aligned} \mathbb{P}(30 \leq \max\{X_1, \dots, X_5\} \leq 32) &= F_{\max\{X_1, \dots, X_5\}}(32) - F_{\max\{X_1, \dots, X_5\}}(30) \\ &= [F_X(32)]^5 - [F_X(30)]^5 \\ &= \left(1 - e^{-\frac{32}{20}}\right)^5 - \left(1 - e^{-\frac{30}{20}}\right)^5 \\ &\simeq 0.3238 - 0.2830 = 0.0408 \end{aligned}$$

Alternative perspective: Try to model this event as “all visits take less than 32 minutes and at least one visit takes more than 30 minutes”.

Exercise 2. Show that each of the following distributions is a member of the exponential family:

- a) Exponential distribution
- b) Geometric distribution

Solution:

Recall that a random variable X is said to belong to the exponential family if its pdf (or pmf) can be expressed as:

$$f(x; \boldsymbol{\lambda}) = h(x)c(\boldsymbol{\lambda}) \exp \left\{ \sum_{i=1}^k \omega_i(\boldsymbol{\lambda}) t_i(x) \right\}$$

a) The Exponential distribution has density:

$$f(x; \lambda) = \lambda \exp\{-\lambda x\} \mathbb{I}_{[0, +\infty)}(x)$$

This belongs to the exponential family with $k = 1$:

$$\begin{aligned} h(x) &= \mathbb{I}_{[0, +\infty)}(x) \\ c(\lambda) &= \lambda \\ \omega_1(\lambda) &= -\lambda \\ t_1(x) &= x \end{aligned}$$

b) The Geometric distribution has probability mass function:

$$f(x; \lambda) = \lambda(1 - \lambda)^x \mathbb{I}_{\mathbb{N}_0}(x) = \lambda \exp \{x \ln(1 - \lambda)\} \mathbb{I}_{\mathbb{N}_0}(x)$$

This belongs to the exponential family with $k = 1$:

$$\begin{aligned} h(x) &= \mathbb{I}_{\mathbb{N}_0} \\ c(\lambda) &= \lambda \\ \omega_1(\lambda) &= \ln(1 - \lambda) \\ t_1(x) &= x \end{aligned}$$

Exercise 3. Let $X_1, \dots, X_n$ be a random sample from a pdf

$$f(x; \lambda) = \frac{1}{\lambda} \exp \left\{ 1 - \frac{x}{\lambda} \right\} \quad \text{with } \lambda > 0 \text{ and } \lambda < x < \infty.$$

Show that it is a probability density function but not a member of the exponential family.

Solution:

To verify that $f(x; \lambda)$ is a probability density function, we evaluate its integral over the support (λ, ∞) :

$$\int_{\lambda}^{\infty} \frac{1}{\lambda} \exp \left\{ 1 - \frac{x}{\lambda} \right\} dx = - \left[\exp \left\{ 1 - \frac{x}{\lambda} \right\} \right]_{\lambda}^{\infty} = - \lim_{x \rightarrow \infty} \exp \left\{ 1 - \frac{x}{\lambda} \right\} + \exp \left\{ 1 - \frac{\lambda}{\lambda} \right\} = 0 + e^0 = 1$$

Since the integral equals 1 and $f(x; \lambda) > 0$, it is a valid pdf.

However, it is not a member of the exponential family because the support of the distribution, $x \in (\lambda, \infty)$, depends explicitly on the unknown parameter λ .

This result might seem counterintuitive at first, since the algebraic component of the pdf appears to factorize cleanly. However, pdfs are defined over all $\mathbb{R}$, requiring an explicit indicator function:

$$f(x; \lambda) = \frac{1}{\lambda} \exp \left\{ 1 - \frac{x}{\lambda} \right\} \mathbb{I}_{(\lambda, \infty)}(x)$$

Because the indicator function $\mathbb{I}_{(\lambda, \infty)}(x)$ couples λ and x non-exponentially, it is impossible to express the density in the form $h(x)c(\lambda) \exp \left\{ \sum_{i=1}^k \omega_i(\lambda)t_i(x) \right\}$. Key takeaway: do not forget the indicator function (as I purposely did in this exercise xD)!

Exercise 4. Let $x_1, \dots, x_n$ be an i.i.d. sample from the Exponential distribution with parameter $\lambda > 0$:

$$f(x; \lambda) = \lambda \exp\{-\lambda x\}, \quad x > 0$$

with $\mathbb{E}(X) = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$.

- Find the statistical model (i.e. the joint probability density function of the sample).
- Show that the statistical model belongs to the exponential family.
- Is the statistic $t(\mathbf{x}) = \sum_{i=1}^n x_i$ sufficient for λ ?
- Is the estimator $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n x_i$ unbiased for λ ?
- Propose an unbiased estimator for the parameter λ .

Solution:

a) Let's start from the joint density of the random sample:

$$\begin{aligned} f_{X_1, \dots, X_n}(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n) &\stackrel{i.i.d.}{=} \prod_{i=1}^n f_{X_i}(x_i; \lambda_i) \\ &\stackrel{i.i.d.}{=} \prod_{i=1}^n f_X(x_i; \lambda) \\ &= \prod_{i=1}^n \lambda \exp \{-\lambda x_i\} \mathbb{I}_{[0, +\infty)}(x_i) \\ &= \lambda^n \exp \left\{ -\lambda \sum_{i=1}^n x_i \right\} \prod_{i=1}^n \mathbb{I}_{[0, +\infty)}(x_i) \end{aligned}$$

b) The joint density can be written in exponential family form for $k = 1$ as:

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n; \lambda) = h(x_1, \dots, x_n) c(\lambda) \exp \{ \omega_1(\lambda) t_1(x_1, \dots, x_n) \}$$

where:

$$\begin{aligned} h(x_1, \dots, x_n) &= \prod_{i=1}^n \mathbb{I}_{[0, +\infty)}(x_i) \\ c(\lambda) &= \lambda^n \\ \omega_1(\lambda) &= -\lambda \\ t_1(x_1, \dots, x_n) &= \sum_{i=1}^n x_i \end{aligned}$$

Hence, the statistical model belongs to the exponential family.

c) According to the Fisher-Neyman Factorization Theorem, a statistic $t(x_1, \dots, x_n)$ is sufficient for λ if $f_{X_1, \dots, X_n}(x_1, \dots, x_n; \lambda) = h(x_1, \dots, x_n) g(t(x_1, \dots, x_n), \lambda)$. Here:

$$\begin{aligned} f_{X_1, \dots, X_n}(x_1, \dots, x_n; \lambda) &= 1 \cdot \left(\lambda^n \exp \left\{ -\lambda \sum_{i=1}^n x_i \right\} \prod_{i=1}^n \mathbb{I}_{[0, +\infty)}(x_i) \right) \\ &= h(x_1, \dots, x_n) g(t(x_1, \dots, x_n), \lambda). \end{aligned}$$

Thus, $t(x_1, \dots, x_n) = \sum_{i=1}^n x_i$ is a sufficient statistic for λ (or, more properly, *for this specific parametric model*).

d) The bias of $\bar{X}_n$ as an estimator for λ is:

$$\text{Bias}(\bar{X}_n) = \mathbb{E}(\bar{X}_n) - \lambda = \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n X_i \right) - \lambda = \frac{1}{\lambda} - \lambda \neq 0$$

So $\bar{X}_n$ is not unbiased for λ .

e) A natural candidate for estimating $\lambda = \frac{1}{\mathbb{E}(X)}$ is the reciprocal of the sample mean, $g(\bar{X}_n) = \frac{1}{\bar{X}_n}$. However, expected values do not commute with non-linear transformations. By Jensen's Inequality, since $g(t) = \frac{1}{t}$ is strictly convex for $t > 0$:

$$\mathbb{E} \left(\frac{1}{\bar{X}_n} \right) > \frac{1}{\mathbb{E}(\bar{X}_n)} = \frac{1}{\frac{1}{\lambda}} = \lambda$$

This reveals that $\frac{1}{\bar{X}_n}$ is biased upward.

To determine the exact bias, define $Y = \sum_{i=1}^n X_i$. Since $X_i \stackrel{\text{iid}}{\sim} \text{Exp}(\lambda) \equiv \text{Gamma}(1, \lambda)$, the sum follows a Gamma distribution: $Y \sim \text{Gamma}(n, \lambda)$ with probability density function $f_Y(y; n, \lambda) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}$ for $y > 0$.

Note: If $Y \sim \text{Gamma}(n, \lambda)$, its reciprocal $W = \frac{1}{Y}$ follows an Inverse-Gamma distribution, $W \sim \text{InvGamma}(n, \lambda)$, with known mean $\mathbb{E}(W) = \frac{\lambda}{n-1}$ for $n > 1$. Computing $\mathbb{E}\left(\frac{1}{Y}\right)$ explicitly:

$$\begin{aligned}\mathbb{E}\left(\frac{1}{Y}\right) &= \int_0^\infty \frac{1}{y} \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} dy \\ &= \frac{\lambda^n}{(n-1)!} \int_0^\infty y^{n-2} e^{-\lambda y} dy\end{aligned}$$

Recognizing the integrand as the kernel of a $\text{Gamma}(n-1, \lambda)$ density, its integral equals $\frac{\Gamma(n-1)}{\lambda^{n-1}} = \frac{(n-2)!}{\lambda^{n-1}}$:

$$\mathbb{E}\left(\frac{1}{Y}\right) = \frac{\lambda^n}{(n-1)!} \cdot \frac{(n-2)!}{\lambda^{n-1}} = \frac{\lambda}{n-1} \quad \text{for } n > 1$$

Since $\frac{1}{\bar{X}_n} = \frac{n}{Y}$, its expectation is:

$$\mathbb{E}\left(\frac{1}{\bar{X}_n}\right) = n \mathbb{E}\left(\frac{1}{Y}\right) = \frac{n}{n-1} \lambda$$

Thus, the bias of $\frac{1}{\bar{X}_n}$ is $\frac{n}{n-1} \lambda - \lambda = \frac{\lambda}{n-1} > 0$. Multiplying by the scaling factor $\frac{n-1}{n}$ corrects the bias, yielding the unbiased estimator:

$$\hat{\lambda} = \frac{n-1}{n \bar{X}_n} = \frac{n-1}{\sum_{i=1}^n X_i}.$$

Additional key takeaways for this specific lecture:

- Do not be scared by the notation! Learn to use λ, θ, ψ and so on interchangeably. Just make sure to stick to a single notation!
- Do not be scared by n -dimensional vectors like $(x_1, \dots, x_n)$! Learn to map complex data structures to objects that you can control. Once you have digested the difference between x, x_i and $(x_1, \dots, x_n)$, you can begin to relax the notation. My personal suggestion is NOT TO RUSH while solving an exercise. I know that writing a thousand times $f_{X_1, \dots, X_n}(x_1, \dots, x_n; \lambda)$ instead of $f(x)$ is tedious, but it is so for a reason!
- Do not be scared by integrals, re-parametrizations or the amount of random variables that you will encounter! Statistics is a beautiful science because it can be explored both from an extremely mechanical perspective and an extremely intuitive perspective. Choose your method accordingly: you can either start from the definitions every single time, or you can manipulate results that you remember from similar scenarios and adapt them to the problems at hand.