Mathematics – Teacher: P. Gibilisco – Probability Test

1. The birthday problem. Suppose that in each year you have 365 days. Consider a group of n randomly chosen people. Define:

$$A_n = \text{in the group there exists a pair with the same birthday.}$$

Trivially $P(A_1) = 0$ and by the pigeon-hole principle $P(A_{366}) = 1$. According to you the smallest n such that $P(A_n) > \frac{1}{2}$ is ...

$$a)183 \qquad b)23 \qquad c)100$$

Solution. We calculate the answer using $P(A_n) = 1 - P(A_n^c)$.
The number of favorable cases is given by

$$365 \cdot 364 \cdots (365 - n + 1) = \frac{365!}{(365-n)!}$$

The number of possible cases is given by

$$\underbrace{365 \cdot 365 \cdots 365}_{n \text{ times}} = 365^n$$

Therefore

$$P(A_n) = 1 - \frac{365!}{(365-n)!365^n}$$

(Please check that $P(A_1) = 0$).
The smallest n such that $P(A_n) > \frac{1}{2}$ is 23 (not an easy calculation).

2. The False positive problem. You want to check if you have a certain disease (say an infection). In the population of your country the 1% is infected. Moreover the doctors say to you that: a) the probability that the test is positive if you really have the disease is 79% (true positive); the probability that the test is positive if you do not have the disease is 10% (false positive). Your test is positive. Which is the probability that you really are infected?

$$a)79\% \qquad b) > 50\% \qquad c) < 8\%$$

Solution. We can formalize the data of the problem as follows.

$$P(D) = \frac{1}{100}$$

$$P(\mathcal{P}|D) = \frac{79}{100} \qquad P(\mathcal{P}|D^c) = \frac{1}{10}$$

that implies

$$P(D^c) = \frac{99}{100}$$

$$P(\mathcal{P}) = P(\mathcal{P}|D) \cdot P(D) + P(\mathcal{P}|D^c) \cdot P(D^c) = \frac{79}{100} \cdot \frac{1}{100} + \frac{1}{10} \cdot \frac{99}{100}$$

and therefore by Bayes formula

$$P(D|\mathcal{P}) = \frac{P(\mathcal{P}|D)P(D)}{P(\mathcal{P})} = \frac{79}{1069} < \frac{80}{1000} = 8\%$$

3. The Monty Hall problem. Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?". Is it to your advantage to switch your choice?

a) It's better to switch   b) It's better not to switch   c) It doesn't matter.

Solution. Define S as the event where you succeed to find the car.

If you don't switch trivially $P(S) = \frac{1}{3}$.

Otherwise suppose that you decide to switch and let A_1 be the event "The car is behind the door No. 1". Then

$$P(S) = P(S \cap A_1) + P(S \cap A_1^c) =$$

$$= P(S|A_1)P(A_1) + P(S|A_1^c)P(A_1^c) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

So it's better to switch.

4. The queue problem. You are in battalion (n soldiers) in a war zone. The commander needs a volunteer for a suicide mission but nobody is willing to propose himself. So, to choose the soldier for the mission, the commander prepares $n-1$ sticks of the same length and one longer stick (he holds the sticks in his hand so that nobody can see the length). The soldier must form a queue to choose the stick. Which position is the safest in the queue?

a) At the beginning. b) At the end. c) It doesn't matter.

Solution. We solve a similar problem. There is an urn with b black balls and w white balls. You pick a ball from the urn and don't look the ball. Now you choose a second ball. What is the probability that the second ball is white?

B_1 = The first ball is black
B_2 = The second ball is black
W_1 = The first ball is white
W_2 = The second ball is white

$$P(W_2) = P(W_2 \cap B_1) + P(W_2 \cap B_1^c) = P(W_2 \cap B_1) + P(W_2 \cap W_1) =$$

$$= P(W_2|B_1)P(B_1) + P(W_2|W_1)P(W_1) = \frac{w}{w+(b-1)} \cdot \frac{b}{b+w} + \frac{w-1}{(w-1)+b} \cdot \frac{w}{b+w} =$$

$$= \frac{w \cdot (w+b-1)}{(w+b-1)\cdot(b+w)} = \frac{w}{b+w} = P(W_1)$$

So the probability doesn't change!