Consumer Theory

November 11, 2024

- Take $L = 2$ and consider an utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ that represents a convex preference relation $\succsim$ in $X = \mathbb{R}^2_+$.

- Assume u is twice continuously differentiable $\mathcal{C}^2$.

- Let $c \in \mathcal{U}$ be an element in the image of u. We construct an *indifference curve*

$$u(x_1, x_2) = c. \tag{1}$$

  that is the locus of all pairs $(x_1, x_2) \in \mathbb{R}^2_+$ that yield the same utility level c to the consumer.

Indifference Curve

- Let $X_1 = \{x_1 \in \mathbb{R}_+ : \exists x_2 \in \mathbb{R}_+ \text{ s.t. } u(x_1, x_2) = c\}$. By construction,
  - X_1 is non–empty;
  - for each $x_1 \in X_1$, since u is quasi-concave there exists a *unique* $x_2 \in \mathbb{R}_+$ such that $u(x_1, x_2) = c$.
- Equation (1) then defines a function $f : X_1 \to \mathbb{R}_+$ such that $u(x_1, f(x_1)) = c$ for all $x_1 \in X_1$.

Indifference Curve and Marginal Rate of Substitution

- Since all pairs of bundles (x_1, x_2) which belong to a given indifference curve yield to the consumer the same level of utility, say c. Then, by totally differentiating (1), we derive that

$$\frac{dx_2}{dx_1} = -\frac{\dfrac{\partial u}{\partial x_1}(x_1, x_2)}{\dfrac{\partial u}{\partial x_2}(x_1, x_2)} \equiv -\mathrm{MRS}_{1,2}(x_1, x_2) \tag{2}$$

The Consumer's Problem

- Assume $\succsim$ is a rational, continuous and locally non-satiated preference relation, and therefore represented by a continuous utility function u (Theorem 1).

- The consumer's problem (henceforth, *UMP*) is then given by

$$\begin{array}{ll} \max & u(x) \\ s.t. & w - p \cdot x \geq 0 \\ & x \geq 0 \end{array}$$

- Does this problem have a solution?

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- Does this problem have a solution?

- For all $p \gg 0$ and $w > 0$, $\mathcal{B}(p, w)$ is closed and bounded.
  - Bounded: if $x \in \mathcal{B}(p, w)$, then $x_i \geq 0$ and $x_i \leq w/p_i$ for each $i \in \{1, \ldots, n\}$.
  - Closed: let $\{x^k\}$ be a converging sequence in $\mathcal{B}(p, w)$. Since $x^k \geq 0$ for all $k \geq 1$, we have that $\lim x^k = x \geq 0$ as well.

    Consider $g_0(x) = w - p \cdot x$, which is a continuous function in x, that is $g_0(x^k)$ converges to $g_0(x)$, notice that $g_0(x^k) \geq 0$ for all k implying that $g_0(x) \geq 0$. Thus, $x \in \mathcal{B}(p, w)$.

Since u is a continuous function and the set $\mathcal{B}(p, w)$ is closed and bounded, by Weierstrass Theorem: *UMP* has a solution for all $p \gg 0$ and $w > 0$.

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The Utility Maximization Problem - UMP

$$\begin{array}{ll} \max & u(x) \\ s.t. & w - p \cdot x \geq 0 \\ & x \geq 0 \end{array}$$

- The solution to this problem $x(p, w)$ is called the *Walrasian demand correspondence (function)*.
- We call $v(p, w) = u(x(p, w))$ the *indirect utility function*.
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Theorem 3

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Walrasian demand correspondence has the following properties:

1. $x(p, w)$ is homogeneous of degree zero in (p, w), i.e. $x(\alpha p, \alpha w) = x(p, w)$ for every $\alpha > 0$;
2. $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$;
3. if $\succsim$ is convex, and $u(.)$ is quasi-concave, then $x(p, w)$ is a convex set. If $\succsim$ is strictly convex, and $u(.)$ is strictly quasi-concave, then $x(p, w)$ is a singleton.

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ii) $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$;

iii) if $\succsim$ is convex, and $u(.)$ is quasi-concave, then $x(p, w)$ is a convex set. If $\succsim$ is strictly convex, and $u(.)$ is strictly quasi-concave, then $x(p, w)$ is a singleton.

Let us prove all three properties.

i) Follows immediately from the fact that $\mathcal{B}(p, w) = \mathcal{B}(\alpha p, \alpha w)$ for all $\alpha > 0$.

ii) $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$. It follows from LNS.
Assume by contradiction that $p \cdot x < w$ at an $x \in x(p, w)$.
By LNS, there exists an $\epsilon > 0$ small enough and a bundle y in an ϵ-neighborhood of x, $||y - x|| \leq \epsilon$, such that $y \succ x$ and $p \cdot y \leq w$.
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This contradicts that $x \in x(p, w)$.

iii-a) Take $u(.)$ is quasi-concave ($\succsim$ convex) and let $x \in x(p, w)$ and $x' \in x(p, w)$ solve UMP.

We have to show that $\alpha x + (1-\alpha)x' \equiv x'' \in x(p, w)$ for every $\alpha \in [0, 1]$.

Since x and x' solve UMP, it must be $u(x) = u(x')$, denote it u^*.

Since $u(.)$ is quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') \geq u^*$.

Since $\mathcal{B}(p, w)$ is a convex set, $x'' \in \mathcal{B}(p, w)$. Indeed both x and x' are in $\mathcal{B}(p, w)$, and since $x'' \equiv \alpha x + (1-\alpha)x'$, it also satisfies $p(\alpha x + (1-\alpha)x') \leq w$.

Hence, x'' is a budget-feasible bundle which yields utility $u(x'') \geq u^*$, therefore is must be $x'' \in x(p, w)$. Therefore, $x(p, w)$ is a convex set.

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iii-b) Take $u(.)$ is strictly quasi-concave ($\succsim$ strictly convex) and assume by contradiction that x, x' with $x \neq x'$ are two solutions to UMP, i.e. $x \in x(p, w)$ and $x' \in x(p, w)$.

Consider $\alpha x + (1-\alpha)x' \equiv x''$ for every $\alpha \in (0, 1)$. Since x and x' solve UMP, it must be $u(x) = u(x')$, denote it u^*.

Since $u(.)$ is strictly quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') > u^*$.

Since $\mathcal{B}(p, w)$ is a convex set, again it holds that $x'' \in \mathcal{B}(p, w)$.

Hence, x'' is a budget-feasible bundle which yields utility $u(x'') > u^*$, hence neither x nor x' can solve UMP. Therefore, $x(p, w)$ must contain only one element. ■

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The Walrasian demand function

- When $\succsim$ is strictly convex, the solution of UPM is called the Walrasian demand function, denote it $x^*(p, w)$.

- Let us now focus on $x^*(p, w)$ for some comparative-statics exercises.

- We discuss wealth effects and price effects.

- Fix the price level at $\bar{p}$, and consider $x^*(\bar{p}, w)$ as a function of w, this is the *Engel curve*.

- Consider how the demand function $x^*(\bar{p}, w)$ changes for different values of wealth, the set of all the values $\{x^*(\bar{p}, w) : w > 0\}$ is the wealth expansion path.

- Holding the price level fixed at $\bar{p}$, take $x^*(\bar{p}, w)$ differentiable. We can compute for each commodity k,

$$\frac{\partial x_k^*(\bar{p}, w)}{\partial w}$$

this is the wealth effect on the demand of good k.

- If $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} \geq 0$, good k is a normal good;
- if $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} < 0$, good k is an inferior good.
- How would the wealth expansion path of a normal good look like? and of an inferior good?