Consumer Theory

November 18, 2024

The indirect utility function $v(p, w)$

Theorem 4

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the indirect utility function has the following properties:

i) $v(p, w)$ is homogeneous of degree zero in (p, w);
ii) $v(p, w)$ is strictly increasing in w and non-increasing in p;
iii) $v(p, w)$ is quasi-convex, that is the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$;
iv) $v(p, w)$ is continuous in p and w.

The indirect utility function

Proof of Theorem 4 - ii)

ii) We prove that $v(p, w)$ is increasing in w.

Take $w' > w$, then $\mathcal{B}(p, w) \subseteq \mathcal{B}(p, w')$.

In particular, if x^* is the optimal bundle at wealth w, then x^* is feasible when wealth is w'. Hence, $v(p, w') \geq u(x^*) = v(p, w)$.

Since $\succsim$ is LNS, there exists x' that $||x' - x^*|| \leq \epsilon$ such that $x' \succ x^*$ and $x' \in \mathcal{B}(p, w')$ when ϵ is small enough, hence $u(x') > u(x^*)$.

Thus, $v(p, w') \geq u(x') > u(x^*) = v(p, w)$.

Using a similar reasoning, show that $v(p, w)$ is non-increasing in p.

The Expenditure Minimization Problem

The Expenditure Minimization Problem

- The basic idea behind the consumer's problem is to choose a bundle that maximizes utility without violating feasibility.

- There is a second way for the consumer to choose a consumption bundle: pick the least costly bundle that yields him a desired utility level.

- This second form of choice is the one we explore now.

The Expenditure Minimization Problem

- Let $\mathcal{U} = \{u(x) : x \in \mathbb{R}^L_+\}$ denote the set of attainable utility levels.
- For each $\overline{u} \in \mathcal{U}$ and $p \gg 0$, the *expenditure minimization problem (EMP)* is

$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t.} & u(x) \geq \overline{u} \\ & x \geq 0 \end{array}$$

- We denote $h(p, \overline{u})$ the solution to this problem, this is the *Hicksian (or compensated) demand correspondence*.
- The value function $e(p, \overline{u}) = p \cdot h(p, \overline{u})$ is called the *expenditure function*.

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- Existence and uniqueness of a solution to *EMP* are guaranteed by $p \gg 0$ and continuity and strict convexity of $\succsim$.

- These are the same conditions that guarantee existence and uniqueness of a solution to *UMP*.

- In what follows, we assume the existence conditions are satisfied.

Duality

- In neoclassical theory, *UMP* and *EMP* are two mirroring ways to look at the same problem.
- On one hand, in *UMP* the consumer seeks to maximize utility given a fixed wealth/expenditure, namely w.
- On the other hand, in *EMP* the consumer seeks to minimize the expenditure necessary to reach a certain utility level, $\bar{u}$.
- We can formally state this intuition.

Duality: implications for the value functions

Theorem 8

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$. Then,

i) if x^* is optimal in *UMP* when wealth is $w > 0$, then x^* is optimal in *EMP* when the required utility is $u(x^*)$. Moreover, the minimized expenditure level in *EMP* is exactly w, that is $p \cdot x^* = w$;

ii) if x^* is optimal in *EMP* at utility $\overline{u} > u(0)$, then x^* is optimal in *UMP* when wealth is equal to $p \cdot x^*$. Moreover, the maximized level of utility in *UMP* is exactly $\overline{u}$, that is $u(x^*) = \overline{u}$.

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2. if x^* is optimal in *EMP* at utility $\bar{u} > u(0)$, then x^* is optimal in *UMP* when wealth is equal to $p \cdot x^*$. Moreover, the maximized level of utility in *UMP* is exactly $\bar{u}$, that is $u(x^*) = \bar{u}$.

Duality

Proof of Theorem 8

We prove $i)$.

i) By contradiction, assume x^* solves UMP, but is not optimal in *EMP* when the required utility is $u(x^*)$.

Then, there must exist an x' such that $p \cdot x' < p \cdot x^*$ and $u(x') \geq u(x^*)$. Since x^* solves *UMP*, $p \cdot x^* \leq w$.

By LNS of $\succsim$, we can find x'' in an ϵ-ball around x', i.e. $||x'' - x'|| \leq \epsilon$, that satisfies $p \cdot x'' < w$ and $x'' \succ x' \leftrightarrow u(x'') > u(x')$.

This implies that $x'' \in \mathcal{B}(p, w)$ and that x^* is not optimal in *UMP*. A contradiction.

Hence, x^* is optimal in *EMP* and the minimized expenditure is $p \cdot x^*$. Since x^* solves *UMP*, it satisfies Walras' law, i.e. $p \cdot x^* = w$.

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Let us prove part *ii*) of Theorem 8.

ii) Observe that since $\overline{u} \geq u(0)$, then $x^* \neq 0$ and $p \cdot x^* > 0$.

By contradiction, assume x^* solves *EMP* but it is not optimal in *UMP* at wealth $p \cdot x^*$.

Then, there must exist an x' such that $p \cdot x' \leq p \cdot x^*$ and $u(x') > u(x^*)$.

Take $x'' = \alpha x'$ with $\alpha \in (0, 1)$. By continuity of $u(.)$, when α is close to 1, $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$.

This implies that x'' is preferred to x^* in *EMP*, since it guarantees the desired utility at lower expenditure. A contradiction.

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Expenditure Minimization Problem: Summary

- For each $\overline{u} \in \mathcal{U} = \{u(x) : x \in \mathbb{R}^L_+\}$ and $p \gg 0$, the *expenditure minimization problem (EMP)* is

$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t.} & u(x) \geq \overline{u} \\ & x \geq 0 \end{array}$$

- The solution to EMP is the *Hicksian (or compensated) demand correspondence*, $h(p, \overline{u})$;

- The value function of EMP is the *expenditure function*, $e(p, \overline{u}) = p \cdot h(p, \overline{u})$.

- EMP is dual to UMP.

Properties of the Hicksian demand $h(p, \overline{u})$

Parallel to what we did for *UMP*, let us now examine the properties of the Hicksian demand and of the expenditure function.

Theorem 5

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Hicksian demand correspondence has the following properties:

i) it is homogeneous of degree zero in prices, i.e. $h(\alpha p, \overline{u}) = h(p, \overline{u}) \; \forall \; \alpha > 0$;

ii) it satisfies no excess utility, i.e. $u(x) = \overline{u}$ for every $x \in h(p, \overline{u})$;

iii) if $\succsim$ is convex, then $h(p, \overline{u})$ is a convex set. If $\succsim$ is strictly convex, then $h(p, \overline{u})$ is single-valued.

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iii) if $\succsim$ is convex, then $h(p, \overline{u})$ is a convex set. If $\succsim$ is strictly convex, then $h(p, \overline{u})$ is single-valued.

The Hicksian demand correspondence

Proof of Theorem 5 - i.)

Let us prove all three properties.

i) Immediate: the optimal bundle x that minimizes $p \cdot x$ also minimizes $\alpha p \cdot x$ for every $\alpha > 0$, subject to the same constraint $u(x) \geq \bar{u}$.

ii) $h(p,w)$ satisfies no excess utility, i.e. $u(x) = \overline{u}$ for every $x \in h(p,\overline{u})$.

Follows from continuity of $u(.)$. Assume by contradiction that $x \in h(p,\overline{u})$ is such that $u(x) > \overline{u}$.

Consider a bundle $x' = \beta x$, with $\beta \in (0,1)$.

By continuity of $u(.)$, when β is close enough to 1, $u(x') \geq \overline{u}$ and $p \cdot x' < p \cdot x$.

Then, $x \notin h(p,\overline{u})$, a contradiction.

Hence, $u(x)$ must be equal to $\overline{u}$ for every $x \in h(p,\overline{u})$.

iii-a) Take $\succsim$ convex and let x, x' be two solutions to EMP, i.e. $x \in h(p, \overline{u})$ and $x' \in h(p, \overline{u})$.

We have to show that $\alpha x + (1-\alpha)x' \equiv x'' \in h(p, \overline{u})$ for every $\alpha \in [0, 1]$.

Since x and x' solve EMP, it must be that $p \cdot x = p \cdot x' = e^*$.

Hence, $p(\alpha x + (1-\alpha)x') = \alpha p \cdot x + (1-\alpha)p \cdot x' = e^*$, that is any convex combination of solutions of EMP is itself expenditure minimizing.

In addition, we also know that $u(x) = u(x') = \overline{u}$ by $ii.$). Since $u(.)$ is quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') \geq \overline{u}$.

Hence, x'' is an expenditure minimizing bundle which yields utility $u(x'')$ not lower than $\overline{u}$, therefore is must be $x'' \in h(p, \overline{u})$, too. $h(p, \overline{u})$ is a convex set.

The Hicksian demand correspondence

Proof of Theorem 5 - iii.)

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The Hicksian demand correspondence

Proof of Theorem 5 - iii.)

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Hence, x'' is an expenditure minimizing bundle which yields utility $u(x'')$ not lower than $\overline{u}$, therefore is must be $x'' \in h(p, \overline{u})$, too. $h(p, \overline{u})$ is a convex set.

The Hicksian demand correspondence

Proof of Theorem 5

iii-b) Take $\succsim$ strictly convex, i.e. $u(.)$ strictly quasi-concave, then $h(p, \overline{u})$ contains a single element.

Prove it!!

Properties of the Expenditure Function $e(p, \overline{u})$

Theorem 6

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the expenditure function has the following properties:

1. it is homogeneous of degree one in prices, i.e. $e(\alpha p, \overline{u}) = \alpha e(p, \overline{u}) \;\forall\; \alpha > 0$;
2. it is strictly increasing in $\overline{u}$ and non-decreasing in p_l for every $l = 1, \ldots, L$;
3. it is concave in p;
4. continuous in p and $\overline{u}$.

The Expenditure Function $e(p, \overline{u})$

Proof of Theorem 6 - i)

Let us prove properties $i) - iii)$.

i) Follows immediately from the fact that $h(p, \overline{u})$ is homogeneous of degree zero in $(p, \overline{u})$. Indeed since $h(\alpha p, \overline{u}) = h(p, \overline{u})$ for all $\alpha \in [0, 1]$, then also $\alpha p \cdot h(\alpha p, \overline{u}) = \alpha p \cdot h(p, \overline{u})$.

ii-a) We prove that $e(p, \overline{u})$ is strictly increasing in $\overline{u}$.

Suppose, by contradiction, that $e(p, \overline{u})$ is not strictly increasing in $\overline{u}$, and let x' and x'' denote optimal consumption bundles for utility levels u' and u'', respectively, with $u'' > u'$ and $p \cdot x' \geq p \cdot x'' > 0$.

Consider a bundle $\tilde{x} = \beta x''$, where $\beta \in (0, 1)$.

By continuity of $u(\cdot)$, there exists a β close enough to 1 such that $u(\tilde{x}) > u'$ and $p \cdot x' > p \cdot \tilde{x}$, a contradiction.

The Expenditure Function $e(p, \overline{u})$

Proof of Theorem 6 - ii)

ii-b) To show that $e(p, \overline{u})$ is non-decreasing in p_l, consider the price vectors p'' and p' such that $p''_l \geq p'_l$ for commodity l, and $p''_k = p'_k$ for all commodities $k \neq l$.

Let x'' be the solution to the *EMP* for prices p''.

Then, $e(p'', \overline{u}) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', \overline{u})$, where the latter inequality follows from the definition of $e(p', \overline{u})$.

iii) To show that $e(p, \overline{u})$ is concave in p, we need to prove that

$$e(\alpha p + (1-\alpha)p', \overline{u}) \geq \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$$

for every p, p' and for every $\alpha \in [0, 1]$.

Denote $p'' \equiv \alpha p + (1-\alpha)p'$ and let $x'' \in h(\alpha p + (1-\alpha)p', \overline{u})$ be a solution to EMP at price p''.

iii) To show that $e(p, \overline{u})$ is concave in p, we need to prove that

$$e(\alpha p + (1-\alpha)p', \overline{u}) \geq \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$$

for every p, p' and for every $\alpha \in [0, 1]$.

Denote $p'' \equiv \alpha p + (1-\alpha)p'$ and let $x'' \in h(\alpha p + (1-\alpha)p', \overline{u})$ be a solution to EMP at price p''.

Then,

$$e(p'', \overline{u}) = p'' \cdot x'' = (\alpha p + (1 - \alpha)p') \cdot x'' =$$

$$\alpha p \cdot x'' + (1 - \alpha)p' \cdot x'' \geq \alpha p \cdot h(p, \overline{u}) + (1 - \alpha)p' \cdot h(p', \overline{u})$$

indeed, x'' is a sub-optimal choice when the prices are either p or p'.

Since $\alpha p \cdot h(p, \overline{u}) + (1 - \alpha)p' \cdot h(p', \overline{u}) = \alpha e(p, \overline{u}) + (1 - \alpha)e(p', \overline{u})$, the chain of inequalities above expresses the concavity of the expenditure function.