Consumer Theory

November 21, 2024

Duality: implications on demand correspondences

Theorem 8 has known implications on the Walrasian and Hicksian demand correspondences for every commodity. For all $p \gg 0$ and $\overline{u} > u(0)$,

$$x_l(p, w) = h_l(p, v(p, w)) \text{ for each commodity } l = 1, \ldots, L$$

$$h_l(p, \overline{u}) = x_l(p, e(p, \overline{u})) \text{ for each commodity } l = 1, \ldots, L.$$

Duality: implications on demand correspondences

Take

$$h_l(p, \overline{u}) = x_l(p, e(p, \overline{u}))$$

for some arbitrary commodity l and consider a change in prices. (see note)

The Hicksian demand $h(p, \overline{u})$ measures the demand that would emerge if we adjust wealth so maintain the consumer at the same level of utility.

This type of compensation is the Hicksian wealth compensation, and explains why $h(p, \overline{u})$ is the compensated demand correspondence.

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The relationship between Hicksian and Walrasian demands, indirect utility and expenditure functions.

More implications of duality

- Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$.
- Assume also that $\succsim$ is strictly-convex, which implies strictly quasi-concave utility function, hence $x(p, w)$ and $h(p, \bar{u})$ identify a unique optimal bundle for UMP and EMP, respectively.
- We start by examining the relationship between the expenditure function and the Hicksian demand.

Hicksian Demand and Expenditure Function

Shepard's Lemma: the relationship between $e(p, \overline{u})$ and $h(p, \overline{u})$

Hicksian Demand and Expenditure functions

Shepard's Lemma

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$, and that $p \gg 0$.
If $e(p, \overline{u})$ is differentiable in p then, for all p and $\overline{u}$, the Hicksian demand is the derivative of the expenditure function with respect to prices, i.e.

$$\frac{\partial e(p, \overline{u})}{\partial p_k} = h_k(p, \overline{u}) \qquad \text{for } \; k = 1, \cdots, L.$$

- The proof is an implication of the Envelope Theorem.
- In EMP, prices are parameters of the min problem.
- The Envelope Theorem tells us that, in an optimization problem, when measuring the first-order effects of a change in the parameters of the problem on the value function, we can disregard any change in the maximizer (minimizer), and only consider the direct effects.

Hicksian Demand and Expenditure functions

Shepard's Lemma: Proof

- Indeed,

$$e(p,\overline{u}) = p \cdot h(p,\overline{u}) = p_1 \cdot h_1(p,\overline{u}) + \cdots + p_L \cdot h_L(p,\overline{u})$$

- Consider commodity k and differentiate $e(p,\overline{u})$ w.r.t. p_k. By the chain rule

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}) + p_1 \frac{\partial h_1(p,\overline{u})}{\partial p_k} + \cdots + p_L \frac{\partial h_L(p,\overline{u})}{\partial p_k}$$

which can be rewritten as

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}) + \sum_{j=1}^{J} p_j \frac{\partial h_j(p,\overline{u})}{\partial p_k}$$

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Since $h(p,\overline{u})$ is optimal, a change in prices has no first-order effect on demand, i.e. $\frac{\partial h_j(p,\overline{u})}{\partial p_k} = 0$, hence on expenditure, and

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}).$$

When applied to the EMP, the direct effect of a change in p_k on the minimal expenditure, measures the variation of the expenditure $e(p,\overline{u})$ at fixed demand $h(p,\overline{u})$.

Roy's Identity: the relationship between $v(p, w)$ and $x(p, w)$

Walrasian Demand and Indirect Utility functions
Roy's Identity

- Let $u^* = v(p^*, w^*)$. By duality, $v(p, e(p, u^*)) = u^*$ for any p.
- Differentiate with respect to p_j and evaluate at $p = p^*$, we get

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} \frac{\partial e(p^*, u^*)}{\partial p_j} = 0.$$

- Shepard's lemma implies that $\frac{\partial e(p^*, u^*)}{\partial p_j} = h_j(p^*, u^*)$, substituting we get

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} h_j(p^*, u^*) = 0.$$

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- Since $w^* = e(p^*, u^*)$ and $h_j(p^*, u^*) = x_j(p^*, w^*)$, we can write

$$\frac{\partial v(p^*, w^*)}{\partial p_j} + \frac{\partial v(p^*, w^*)}{\partial w} x_j(p^*, w^*) = 0.$$

which gives

$$x_j(p^*, w^*) = -\frac{\dfrac{\partial v(p^*, w^*)}{\partial p_j}}{\dfrac{\partial v(p^*, w^*)}{\partial w}}.$$

Walrasian Demand and Indirect Utility functions

Roy's Identity

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$.
Suppose that $v(p, w)$ is differentiable at $(p^*, w^*) \gg 0$. Then, for every $j = 1, \ldots, L$

$$x_j(p^*, w^*) = -\frac{\dfrac{\partial v(p^*, w^*)}{\partial p_j}}{\dfrac{\partial v(p^*, w^*)}{\partial w}}$$

Walrasian Demand and Indirect Utility functions

- Roy's identity is the analog of Shepard's lemma for the Walrasian demand function.

- When deriving the Walrasian demand from the indirect utility, we have to normalize the price derivative of the indirect utility by the derivative of $v(p, w)$ w.r.t. wealth;

$$x_j(p^*, w^*) = -\frac{\frac{\partial v(p^*, w^*)}{\partial p_j}}{\frac{\partial v(p^*, w^*)}{\partial w}}.$$

- Indeed, utility is an ordinal concept, so is the Walrasian demand, which is then sensitive to the underlying $u(.)$.

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Slutsky Equation: the relationship between Hicksian and Walrasian demands

Hicksian and Walrasian demand functions

- Fix $\bar{u} = v(\bar{p}, w)$ for some $\bar{p} \gg 0$ and $w > 0$.
- By duality $w = e(\bar{p}, \bar{u})$.
- Duality also implies that for all p and u and for each commodity $l = 1, \ldots, L$

$$h_l(p, u) = x_l(p, e(p, u)) \tag{7}$$

- Differentiate both sides of (7) with respect to p_k and evaluate it at $(\bar{p}, \bar{u})$ to get

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}. \tag{8}$$

Hicksian and Walrasian demand functions

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- By Shepards' Lemma, $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$, hence

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- By duality $e(\bar{p},\bar{u}) = w$ and $h_k(\bar{p},\bar{u}) = x_k(\bar{p},e(\bar{p},\bar{u})) = x_k(\bar{p},w)$, thus $(8')$ can be rewritten as:

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The Slutsky Equation

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$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w) \qquad \text{for all } l, k.$$

The Slutsky Equation

- Let $l = k$, the Slutsky equation tells us that

$$\frac{\partial h_l(p,u)}{\partial p_l} = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} x_l(p,w).$$

- For a given commodity, the Slutsky equation relates the slopes of the Hicksian and of the Walrasian demands.
- If commodity l is normal, the Hicksian demand is steeper (more rigid) than the Walrasian demand.
- Indeed, if the price of commodity l increases and its demand falls, the consumer's expenditure increases to guarantee the same level of utility. If such wealth compensation is absent, as in the Walrasian demand, the fall of the demand for commodity l is more pronounced.

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The Slutsky Substitution Matrix

- The matrix that collects all the cross-price derivatives of the Hicksian demands for each commodity l, i.e. $\frac{\partial h_l(p,u)}{\partial p_k}$ for each k, l, is indeed the *Slutsky substitution matrix*, $S(p,w)$.

- Since $S(p,w)$ is obtained by taking the price derivative of the Hicksian demand for each commodity, when demand is generated from EMP, the matrix $S(p,w)$ inherits some properties of the Hicksian demand and of the expenditure function.

The Slutsky Substitution Matrix

Specifically,

- $S(p, w)$ is negative semidefinite [because of Shepard's Lemma and the concavity of $e(p, \overline{u})$];
- $S(p, w)$ is symmetric [i.e. the compensated cross-price derivatives of any two commodities, l and k, are equal, i.e. $\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_l}$];
- $S(p, w)$ is such that $S(p, w) \cdot p = 0$ [by homogeneity of degree zero of $h(p, \overline{u})$].

The property that $S(p, w) \cdot p = 0$, together with the compensated law of demand imply that every commodity has at least one substitute, i.e. since for commodity k, $\frac{\partial h_k(p,u)}{\partial p_k} \leq 0$ there must exist a commodity, say j, such that $\frac{\partial h_j(p,u)}{\partial p_k} \geq 0$.

The Slutsky Equation

The Slutsky equation can be rewritten as follows:

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{TE} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{SE} \underbrace{- \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{IE}$$

The total effect (TE) of a change in price on consumer's demand can be decomposed into two effects: the substitution effect (SE) and the income effect (IE).

SE gives a measure of the effect that a change in price induces in the consumers' demand when wealth is adjusted so to keep the consumer at the same utility level.

IE measures the effect of the same change on the purchasing power of the consumer hence on its Walrasian demand.

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Duality

Roadmap in Duality