Microeconomics I 2024-25

Producer Theory

November 25, 2024

Introduction

- We studied the demand side of the economy.
- In particular, how by disciplining preference relations we are led to an equivalent representation through the utility functions, and from there to consumer's demand, as the result of an optimal choice over a set of feasible (affordable) alternatives.
- Now, we look at the supply side of the economy.
- The economic agent of interest now is the *firm*.

Introduction

- Assume the firm is atomistic and non–strategic.
- No specific assumption is made on the market structure in which the firm operates at this stage.
- Being atomistic and non-strategic, it takes the market prices as given, for the production good as well as for the inputs.

Production Technology

Primitives

- Suppose there are m goods in the economy.
- A *production plan* is a vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, in which
  - $y_i > 0$ means good i is an *output*, and y_i is the amount produced;
  - $y_i < 0$ means good i is an *input*, and $-y_i$ is the amount used as input.
- A production plan is feasible if it belongs to the *production set* Y, that is $y \in Y$. Y is exogenously given.

The Production Function

Example with a single output

- When we want to represent a firm that produces a single output, the production set Y is described by means of the *production function*, which represents the technology that uses inputs to produce the output.

- Then, a production plan is $y = (q, z_1, \ldots, z_{m-1}) \in \mathbb{R}^m$ in which q is the output, and $(z_1, \ldots, z_{m-1})$ represent the amount of goods that can only be used as inputs.

- $f : \mathbb{R}_+^{m-1} \to \mathbb{R}_+$ is the *production function*, which yields the maximum amount of output that can be obtained combining the inputs $(z_1, \ldots, z_{m-1})$ according to the technology $f(.)$

- Let the production function f be differentiable. Fix the output level at $\overline{q} = f(\overline{z})$ and consider two inputs l, k.

- The marginal rate of technical substitution of input l for input k at $\overline{z}$ is

$$MRTS_{lk}(\overline{z}) = -\frac{\frac{\partial f(\overline{z})}{\partial z_l}}{\frac{\partial f(\overline{z})}{\partial z_k}}$$

- $MRTS_{lk}(\overline{z})$ measures by how much the use of input k has to increase, if we decrease the use of input l by one marginal unit, to keep the level of output at $\overline{q}$.

- $MRTS_{lk}(\overline{z})$ measures the slope of the production function at point $\overline{z}$.

Properties of the production set $Y = \{y \in \mathbb{R}^m : q - f(z_1, \cdots, z_{m-1}) \leq 0\}$.

1. Y is non – empty.
2. Y is closed.
3. Possibility of inaction: $0 \in Y$.
    - The firm can shut down completely. This is possible if there are no sunk costs, i.e., no commitments to the use of some inputs for production.

Properties of the production set $Y = \{y \in \mathbb{R}^m : q - f(z_1, \cdots, z_{m-1}) \leq 0\}$.

i) Y is non – empty.

ii) Y is closed.

iii) Possibility of inaction: $0 \in Y$.

- The firm can shut down completely. This is possible if there are no sunk costs, i.e., no commitments to the use of some inputs for production.

Properties of the production set $Y = \{y \in \mathbb{R}^m : q - f(z_1, \cdots, z_{m-1}) \leq 0\}$.

i) Y is non – empty.

ii) Y is closed.

iii) Possibility of inaction: $0 \in Y$.

- The firm can shut down completely. This is possible if there are no sunk costs, i.e., no commitments to the use of some inputs for production.

iv) No Free Lunch: $Y \bigcap \mathbb{R}^m_+ \subseteq \{0\}$.

- This implies that there is no $y > 0$ in Y. In other words, there is no feasible production plan where some goods are produced with no inputs.

v) Free Disposal: $Y - \mathbb{R}^m_+ \subseteq Y$.

- This implies that if $y \in Y$ and we take any $y' \leq y$, then $y' \in Y$. A production plan $y' \leq y$ implies that we can produce lower or equal quantities of outputs as in y with at least as much of the inputs in y.

vi) Irreversibility: if $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

- This implies that switching the roles of inputs and outputs in a production plan is never feasible.

iv) No Free Lunch: $Y \bigcap \mathbb{R}^m_+ \subseteq \{0\}$.

- This implies that there is no $y > 0$ in Y. In other words, there is no feasible production plan where some goods are produced with no inputs.

v) Free Disposal: $Y - \mathbb{R}^m_+ \subseteq Y$.

- This implies that if $y \in Y$ and we take any $y' \leq y$, then $y' \in Y$.
A production plan $y' \leq y$ implies that we can produce lower or equal quantities of outputs as in y with at least as much of the inputs in y.

vi) Irreversibility: if $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

- This implies that switching the roles of inputs and outputs in a production plan is never feasible.

iv) No Free Lunch: $Y \bigcap \mathbb{R}^m_+ \subseteq \{0\}$.

- This implies that there is no $y > 0$ in Y. In other words, there is no feasible production plan where some goods are produced with no inputs.

v) Free Disposal: $Y - \mathbb{R}^m_+ \subseteq Y$.

- This implies that if $y \in Y$ and we take any $y' \leq y$, then $y' \in Y$.
  A production plan $y' \leq y$ implies that we can produce lower or equal quantities of outputs as in y with at least as much of the inputs in y.

vi) Irreversibility: if $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

- This implies that switching the roles of inputs and outputs in a production plan is never feasible.

vii) Non–Increasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \in [0, 1]$.

- Every production plan can be scaled down up to inaction/complete shut down.

viii) Non–Decreasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 1$.

- Every production plan can be scaled up.

ix) Constant Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 0$.

- This combines non-increasing and non-decreasing returns to scale.
- When the returns to scale are constant, the production set is a cone.

vii) Non–Increasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \in [0, 1]$.

- Every production plan can be scaled down up to inaction/complete shut down.

viii) Non–Decreasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 1$.

- Every production plan can be scaled up.

ix) Constant Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 0$.

- This combines non-increasing and non-decreasing returns to scale.
- When the returns to scale are constant, the production set is a cone.

vii) Non–Increasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \in [0, 1]$.

- Every production plan can be scaled down up to inaction/complete shut down.

viii) Non–Decreasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 1$.

- Every production plan can be scaled up.

ix) Constant Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 0$.

- This combines non-increasing and non-decreasing returns to scale.
- When the returns to scale are constant, the production set is a cone.

x) Convexity: Y is convex.

- The production set is convex if for every pair of plans $y, y' \in Y$, $\alpha y + (1-\alpha)y' \in Y$ for every $\alpha \in [0,1]$.
- Notice that if inaction is possible, then convexity implies non–increasing returns to scale.
  Indeed, if Y is convex, then $\alpha y = \alpha y + (1-\alpha)0 \in Y$ for all $\alpha \in [0,1]$ and for every $y \in Y$.

x) Convexity: Y is convex.

- The production set is convex if for every pair of plans $y, y' \in Y$, $\alpha y + (1-\alpha)y' \in Y$ for every $\alpha \in [0,1]$.

- Notice that if inaction is possible, then convexity implies non–increasing returns to scale.
  Indeed, if Y is convex, then $\alpha y = \alpha y + (1-\alpha)0 \in Y$ for all $\alpha \in [0,1]$ and for every $y \in Y$.

x) Convexity: Y is convex.

- The production set is convex if for every pair of plans $y, y' \in Y$, $\alpha y + (1-\alpha)y' \in Y$ for every $\alpha \in [0,1]$.
- Notice that if inaction is possible, then convexity implies non–increasing returns to scale.
  Indeed, if Y is convex, then $\alpha y = \alpha y + (1-\alpha)0 \in Y$ for all $\alpha \in [0,1]$ and for every $y \in Y$.

Single-output Production Technology and convexity

1. We now show that for the single-output case, Y is convex if and only if $f(z)$ is a concave production function.

xi) Additivity: $Y + Y \subseteq Y$.

- Additivity implies that for every $y \in Y$ and $y' \in Y$, the plan $y + y' \in Y$.
- It is a weaker version of non–decreasing returns to scale, which is called *free entry*. Free entry implies that if y is feasible, then replicating y twice is also feasible, i.e., if $y \in Y$ then $2y \in Y$
  Now change 2 into any positive integer k, you can then consider an arbitrary number of firms who can possibly become active in the market you are considering.

xi) Additivity: $Y + Y \subseteq Y$.

- Additivity implies that for every $y \in Y$ and $y' \in Y$, the plan $y + y' \in Y$.

- It is a weaker version of non–decreasing returns to scale, which is called *free entry*. Free entry implies that if y is feasible, then replicating y twice is also feasible, i.e., if $y \in Y$ then $2y \in Y$
  Now change 2 into any positive integer k, you can then consider an arbitrary number of firms who can possibly become active in the market you are considering.

- Observe that the properties of Convexity and of Constant Returns to Scale imply that Y is a convex cone.

The production set is a convex cone if for every pair of plans $y, y' \in Y$ and constants $\alpha \geq 0$ and $\beta \geq 0$, then

$$\alpha y + \beta y' \in Y.$$

One can show that.

Proposition 1

The production set is additive and satisfies non-increasing returns to scale if and only if it is a convex cone.

Proof of Proposition 1.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is

  for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

  $$\alpha y + \beta y' \in Y$$

- Pick a pair $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right) ky \in Y$ by non-increasing returns to scale.
- Similarly for $\beta y'$, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right) ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

Proof of Proposition 1.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is

  for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

$$\alpha y + \beta y' \in Y$$

- Pick a pair $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right) ky \in Y$ by non-increasing returns to scale.
- Similarly for $\beta y'$, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right) ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

Proof of Proposition 1.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is

  for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

$$\alpha y + \beta y' \in Y$$

- Pick a pair $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right) ky \in Y$ by non-increasing returns to scale.
- Similarly for $\beta y'$, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right) ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

Proof of Proposition 1.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is

  for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

$$\alpha y + \beta y' \in Y$$

- Pick a pair $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right) ky \in Y$ by non-increasing returns to scale.
- Similarly for $\beta y'$, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right) ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

Proof of Proposition 1.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is

  for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

$$\alpha y + \beta y' \in Y$$

- Pick a pair $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right) ky \in Y$ by non-increasing returns to scale.
- Similarly for $\beta y'$, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right) ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

Proof of Proposition 1.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is

  for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

$$\alpha y + \beta y' \in Y$$

- Pick a pair $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right) ky \in Y$ by non-increasing returns to scale.
- Similarly for $\beta y'$, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right) ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

- One way to think about the assumption of convexity of the production set is that it is an implication of additivity, i.e. the possibility to replicate feasible production plans in bigger scale without creating negative interferences to the production activity, and of non-increasing returns to scale, i.e. the possibility to always scale down any input-output combination.

Profit Maximization

- Examine the market behavior of an individual firm.
- The firm takes as given the market price of each good, $(p_1, \ldots, p_m)$.
- We consider that the objective of the firm is to maximize profits.

- Let $p = (p, w_1, \ldots, w_{m-1}) \gg 0$ be the vector of prices of the unique output good p and of the $(m-1)$ inputs w_i with $i = 1, ..., m-1$.

- The profit maximization problem (PMP) of the firm is then

$$\max_{q,z\geq 0} \; p\,q - w \cdot z$$
$$s.t. \; q - f(z) \leq 0$$

- The constraint set in this problem is closed and bounded, when we add the condition of free disposal, the *PMP* has a solution.

Profit Maximization

Lemma 1

If Y is closed, bounded and satisfies free-disposal, the profit maximization problem *(PMP)* has a solution.

- As with UMP for the consumer, we name the solution to and the value function of the profit maximization problem:

  the *supply correspondence*,

  $$y(p) = \{(q, z) \in \mathbb{R}^m : q - f(z) \leq 0 \textit{ and the profit is maximum}\}$$

  and the *profit function*,

  $$\pi(p) = \max\{pq - w \cdot z : q - f(z) \leq 0\}.$$

- The optimizer of PMP gives to the firm the highest possible profits, given the technological constraint $(f(y))$.

- It corresponds to a point which simultaneously belongs to an iso-profit curve and to the production set.

- Specifically, any production plan that solves PMP, $y^* \in y(p)$, is a point at which the highest iso-profit curve of the firm is tangent to the boundary of the production set, identified by the production function.

The Profit Maximization Problem (PMP)

- Given $p = (p, w_1, \ldots, w_{m-1}) \gg 0$, the profit maximization problem (PMP) of the firm is

$$\begin{aligned} \max_{q,z} \;\; & p\, q - w \cdot z \\ s.t. \;\; & q - f(z) \leq 0 \\ & z \geq 0 \end{aligned}$$

Profit Maximization: single output

- Suppose $f(.)$ is a differentiable function, the necessary first order conditions for z^* to solve this optimization problem are

$$p \frac{\partial f(z^*)}{\partial z_k} \leq w_k \qquad \text{for every } k = 1, \ldots, m-1 \tag{1}$$

with (1) holding as an equality if $z_k^* > 0$.

Profit Maximization: single output

- The first order conditions in (1) imply that at the solution z^*, for every pair of inputs l, k

$$MRTS_{l,k}(z^*) = \frac{w_l}{w_k}$$

- That is, at the optimum, the marginal rate of technical substitution be equal to the ratio of input prices, which is the economic rate of substitution, for every pair of inputs.

- If Y is convex, the first-order conditions are *necessary and sufficient* for identifying the maximum production plan.

Profit function and supply correspondence: properties

Theorem 1

Suppose that Y is non-empty, closed and satisfies free-disposal. Let $\pi(p)$ be the profit function and $y(p)$ the supply correspondence of the profit maximization problem for Y, then

i) $\pi(p)$ is homogeneous of degree one, i.e. $\pi(\alpha p) = \alpha\pi(p)$ for every $\alpha > 0$;

ii) $\pi(p)$ is convex;

iii) if Y is convex, then $Y = \{y \in \mathbb{R}^m : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$;

iv) $y(p)$ is homogeneous of degree zero;

Profit function and supply correspondence: properties

Theorem 1- continued

v) if Y is convex, $y(p)$ is a convex set for all p. If Y is strictly convex, $y(p)$ is a singleton;

vi) $[Hotelling's\ lemma]$ If $y(\bar{p})$ is a singleton, then $\pi(.)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$;

vii) If $y(\bar{p})$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semi-definite matrix, with $Dy(\bar{p}) \cdot \bar{p} = 0$.

i) Homogeneity of degree one of $\pi(p)$ follows immediately from the linearity of the profit function.

iii) $\pi(p)$ gives a way to characterize the technology through the profit function.

iv) If all the input prices and the output one vary by the same proportion, the firm's optimal demands of inputs and supplies of outputs do not change ($y(p)$ is homogeneous of degree zero).

- $\pi(p)$ convex in p implies that, for every pair (p, p') and $\alpha \in [0, 1]$,

$$\pi(\alpha p + (1-\alpha)p') \leq \alpha\pi(p) + (1-\alpha)\pi(p')$$

- Take two price profiles, p and p' and supply correspondences, $y(p)$ and $y(p')$, respectively. Consider their convex combination $\alpha p + (1-\alpha)p'$, and an optimal production plan at that combination, $\tilde{y} \in y(\alpha p + (1-\alpha)p')$

- Evaluate the value of the profit function at $\tilde{y}$, $\pi(\alpha p + (1-\alpha)p') = (\alpha p + (1-\alpha)p')\tilde{y} = \alpha p\tilde{y} + (1-\alpha)p'\tilde{y} \leq \alpha p y(p) + (1-\alpha)p' y(p') = \alpha\pi(p) + (1-\alpha)\pi(p')$. ■

v) Uniqueness (multiplicity) result parallels what we found for the consumer problem(s); here it depends on the convexity of Y.

vi) Same idea of the Shephard's Lemma.

vii) The positive semi-definiteness of $D^2\pi(p)$ identifies the law of supply, i.e., the own price effects on the (net) supply correspondence of each good are non-negative. That is, if the price of an output increases, its (optimal) supply increases, and if the price of an input increases, its (optimal) demand decreases.

Cost Minimization

- Notice that when solving PMP the firm looks for the least costly combination of the inputs that allows to produce the optimal levels of output goods, given her technology

- Hence, as an implication of choosing the profit-maximizing production plan, the neoclassical firm minimizes her costs of production

- Consider the single-output case, and formulate the cost-minimization problem of the firm when p is the price of the output, and $w = (w_1, \cdots, w_{m-1})$ is the vector of the $(m-1)$ input prices

Cost Minimization Problem: single-output case

- Fix the level of output at q, and consider the following problem

$$\begin{array}{rl} \max_z & -\left(w_1 z_1 + ... + w_{m-1} z_{m-1}\right) \\ s.t. & f\left(z_1, ..., z_{m-1}\right) \geq q \end{array}$$

- This is equivalent to

$$\begin{array}{rl} \min_z & \left(w_1 z_1 + ... + w_{m-1} z_{m-1}\right) \\ s.t. & f\left(z_1, ..., z_{m-1}\right) \geq q \end{array}$$

Cost Minimisation Problem: single-output case

- At fixed q, the cost minimisation problem (CMP) is

$$\begin{array}{ll} \max_z & -(w_1 z_1 + \ldots + w_{m-1} z_{m-1}) \\ s.t. & f(z_1, \ldots, z_{m-1}) \geq q \end{array}$$

- The solution to CMP is a profile of inputs $(z_1^*, \ldots, z_{m-1}^*)$ that minimizes the firm's cost of production, for given q and input prices $(w_1, \ldots, w_{m-1})$

- Each $z_i^*(w, q)$ is the *conditional factor demand* for input i at prices w and production level q

- The value of the problem is the *cost function*, labelled $C(w, q)$.

Cost Minimization: single output

Characterization by Lagrange method

- Suppose $f(.)$ is differentiable, the necessary first order conditions for a solution to such optimization problem are

$$w_k \geq \lambda \frac{\partial f(z^*)}{\partial z_k} \qquad \text{for every } k = 1, \dots, m-1 \tag{2}$$

with equality if $z_k^* > 0$

- If Y is convex, i.e., $f(.)$ is concave, these first-order conditions are *necessary and sufficient* for identifying the cost-minimasing vector of inputs.

Cost Minimization: single output

Optimality conditions

- As for the PMP, the first order conditions in (2) imply that at the solution z^*, for every pair of inputs l, k

$$MRTS_{l,k}(z^*, q) = \frac{w_l}{w_k}$$

- That is, at the optimum, the marginal rate of technical substitution be equal to the ratio of input prices, which is the economic rate of substitution, for every pair of inputs

- The marginal rate of technical substitution is the slope of the isoquant associated to the production level q

- Interpret the Lagrange multiplier as the marginal value of relaxing the constraint of CMP, i.e., $f(z) \geq q$

- Specifically, λ now measures the marginal cost of production, $\frac{\partial C(w,q)}{\partial q}$ (See proof in the Note)

Cost function and conditional factor demands: properties

Theorem 2

Suppose that Y is non-empty, closed and satisfies free-disposal. Let $C(w,q)$ be the cost function and $z(w,q)$ be the conditional factor demands for the $(m-1)$ inputs of the cost minimization problem for a single-output technology Y with production function f, then

i) $C(w,q)$ is homogeneous of degree one in w, and non-decreasing in q;

ii) $C(w,q))$ is concave in w;

iii) if the set $\{z \geq 0 : f(z) \geq q\}$ is convex for every q, then $Y = \{(-z,q) \in \mathbb{R}^m : w \cdot z \geq C(w,q) \text{ for all } w \gg 0\}$;

iv) $z(w,q)$ is homogeneous of degree zero in w;

Cost function and conditional factor demands: properties

Theorem 2- continued

v) if the set $\{z \geq 0 : f(z) \geq q\}$ is convex, $z(w, q)$ is a convex set. If the set $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, $z(w, q)$ is a singleton;

vi) $[Shepard's\ lemma]$ If $z(w, q)$ is a singleton, then $C(.)$ is differentiable at $\bar{w}$ and $\nabla_w C(\bar{w}, q) = z(\bar{w}, q)$;

vii) If $z(w, q)$ is a function, differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D^2_w C(\bar{w}, q)$ is a symmetric and negative semi-definite matrix, with $D_w z(\bar{w}, q) \cdot \bar{w} = 0$;

Cost function and conditional factor demands: properties

Theorem 2- continued

viii) If $f(.)$ is a homogeneous of degree one, i.e., it exhibits constant returns to scale, then $z(.)$ and $C(.)$ are homogeneous of degree one in q;

ix) If $f(.)$ is concave, then $C(.)$ is a convex function of q, and the marginal costs are non-decreasing in q.

i) Total costs linearly increase in input prices.

iii) It is a way to characterize the technology through the cost function, parallel to PMP, and it builds a dual relationship between costs and profits.

iv) If all the input prices vary by the same proportion, the firm's optimal demands of inputs do not change.

Cost function and conditional factor demands: properties

Proof of Theorem 2

v) Parallel to what we found for the consumer problem(s), and for PMP, stated for the relevant production set.

vi) Same idea of the Shephard's Lemma in consumption.

vii) The negative semi-definiteness of $D_w^2 \pi(p)$ identifies the law of supply for the conditional factor demands, , i.e., if the price of an input increases, its (optimal) demand decreases.

Profit Maximization with cost function: single output

- Once the cost minimization problem has been solved, the firn has to determine the optimal level of output q, by solving the following problem

$$\max_q pq - C(w, q)$$

- The solution of this problem is a level of q^* that maximizes profits while minimizing the firm's cost of production, given the technology $f(.)$ and the input prices $(w_1, ..., w_{m-1})$

Profit Maximization with cost function: single output

- The necessary first order conditions for a solution to such optimization problem are

$$p - \frac{\partial C(w,q)}{\partial q} = 0 \tag{3}$$

  with equality if $q > 0$

- At an interior optimum, price equal marginal cost.

- If $C(w,q)$ is convex in q, i.e., $f(.)$ is concave, this first-order condition is *necessary and sufficient* for identifying q^*, i.e. the firm's optimal output level

Aggregate Supply

- Suppose there are J firms in the economy, each with a given production set $Y_1, \ldots, Y_J$, which is non-empty, closed and satisfies free disposal

- The aggregate supply correspondence is the sum of the individual supplies of each firm

- The aggregate profits obtained when each firm maximizes profits independently are the same that would be obtained if the firms were coordinating their actions in a joint profit maximization decision

Efficiency

- Are the optimal production plans of a neoclassical firm efficient?
- A production plan $y \in Y$ is *efficient* is there exists no other production plan $\tilde{y} \in Y$, such that $\tilde{y} \geq y$ and $\tilde{y} \neq y$
- In other words, for a production plan to be efficient there must be no other feasible plan that either generates the same output with less inputs, or that produces more output with the same inputs
- In a single-output model of production, efficient plans are those for which $f(z_1, ..., z_{m-1}) = q$.

First Theorem of Welfare Economics

First Welfare Theorem

If $y \in Y$ is profit maximizing for some $p \gg 0$, then it is efficient.

Proof. Suppose by contradiction that there is a $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. Since all prices (output and input) are strictly positive this implies that $p \cdot y' > p \cdot y$, which implies that y is not profit maximizing. ■

First Theorem of Welfare Economics

First Welfare Theorem

If $y \in Y$ is profit maximizing for some $p \gg 0$, then it is efficient.

Proof. Suppose by contradiction that there is a $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. Since all prices (output and input) are strictly positive this implies that $p \cdot y' > p \cdot y$, which implies that y is not profit maximizing. ■

First Theorem of Welfare Economics: comments

- The first welfare theorem implies that if many (atomistic) firms independently maximize profits at prices $p \gg 0$, the aggregate result is efficient for the economy
- Having strictly positive prices is essential, otherwise it is possible to find an example of a production plan $y \in Y$ that is profit-maximizing for some $p \geq 0$ (with $p \neq 0$) which is socially inefficient!

Second Theorem of Welfare Economics

Second Welfare Theorem

Assume Y is a convex set. Then, every efficient production plan $y \in Y$ is profit-maximizing for some non-zero price vector $p \geq 0$.

Proof. Suppose that $y \in Y$ is efficient. Define the set $P_y = \{y' \in \mathbb{R}^m : y' \gg y\}$. P_y is a convex set and because y is efficient (hence, on the frontier of the transformation function), $P_y \cap Y = \emptyset$.

- Using the separating hyperplane theorem, we know that there is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$
- This implies in particular, that $p \cdot y' \geq p \cdot y$ for every $y' \gg y$
- Therefore, we must have $p \geq 0$ because if $p_k < 0$ for some k, we would have $p \cdot y' < p \cdot y$ for $y' \gg y$ with $y'_k - y_k$ sufficiently large
- Now take any $y'' \in Y$. Then, $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$.
- Because we can choose y' arbitrarily close to y, we can conclude that $p \cdot y \geq p \cdot y''$ for every $y'' \in Y$, that is, y is profit maximizing. ■