Advanced Microeconomics - MSc Economics

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Università degli Studi di Roma Tor Vergata

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Introduction

- Microeconomics deals with the analysis of economic choices. Economic agents are consumers, firms, financial intermediaries, who operate in different contexts.
- Consumers typically take consumption and saving decisions, firms deal with investment and production decisions, financial intermediaries handle financial decisions of consumers and firms and manage their own portfolios... all of them interact in (complex) economic systems.
- As a result of their choices, market demand and supply emerge, to which economists combine the notion of equilibrium, typically in terms of price and quantity.

- This course offers a formal (logico-mathematical) approach to the analysis of the choices of consumers and firms operating in market economies and deals with the notion of equilibrium in competitive markets.
- When examining choices, we will pay attention to individual decision making in setting without and with risky alternatives.
- Eventually, we will discuss the two fundamental Theorems of Welfare Economics for competitive market systems which require a general equilibrium perspective.

Introduction

- References:
  - Mas Colell A., Whinston M. and J. Green, Microeconomic Theory, Oxford University Press
  - Varian H. R., Intermediate Microeconomics: A Modern Approach, Norton & Co
- Contacts

  Eloisa Campioni, office hours on Tuesdays after class, to be arranged via e-mail: eloisa.campioni@uniroma2.it

Structure of the course

- Lectures and practices. Practical classes will be held on Fridays, 9-11.
- Problem sets will be assigned on a regular basis and corrected in class during the practices.

Evaluation

- Each student's final evaluation consists of the combination of the following three elements.

1. *Problem sets.* Each student will hand in her/his solutions to the assigned problem sets, cooperative work is encouraged. The student's solutions will be evaluated/graded and will contribute to 30% of the final grade. The exercises/questions will be corrected during the practices.
2. *Final written exam.* Written closed-book exam (questions and exercises), yields the remaining 70% of the final grade.
3. *In class participation.* Active participation during lectures will also be part of the evaluation. During the practical classes, students will be randomly asked to present their solutions to the assigned problem sets.

Primitives

- In most of this course, we focus on individual economic agents, and make two assumptions about these agents:

  1. *Atomistic:* the agents are small enough compared to the size of the market that their choices do not affect the market price.

  2. *Non–strategic:* agents do not interact when making their choices.

- We start by examining the choice problem of a consumer.

Primitives

- There are four building blocks in modeling consumer choice:
  - *Consumption (Choice) Set:* The set X of all alternatives (complete consumption plans) that the consumer can conceive;
  - *Feasible Set:* The subset B of X that is achievable given the constraints the consumer faces;
  - *Consumer's Preferences:* A rule specifying how the consumer ranks different alternatives;
  - *Behavioral Assumption:* The consumer seeks to identify a feasible alternative that is preferred to all other feasible alternatives.

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Consumption set

- Consider an economy with L commodities. Assume that the consumption set X satisfies:
    - X is non-empty, specifically $X = \mathbb{R}^L_+$;
    - X is closed;
    - X is convex;
    - $\mathbf{0} \in X$;
    - consumption of larger quantities is always feasible, i.e. if $x \in X$ and $y \geq x$, then $y \in X$.
- A typical element of X is denoted by $x = (x_1, \ldots, x_L)$, where $x_i \geq 0$ is the amount consumed of good $i = 1, 2, ..., L$.

Walrasian/competitive budget set

- Economic constraints on alternatives: consumer cannot achieve what she cannot afford. Some alternatives may not be (economically) feasible. The *Budget set* identifies the set of economically feasible alternatives.

- For economic decisions, feasibility concerns prices and wealth.

Walrasian/competitive budget set

- Let $p = (p_1, \ldots, p_L) \gg 0$ be the vector of prices of L commoditites, and $w > 0$ the consumer's wealth.

- The *budget set* is given by

$$\mathcal{B}(p, w) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}.$$

  with $p \cdot x \ = p_1 x_1 + ... + p_L x_L$.

- The consumer's problem, given price vector p and wealth w, is then to choose $x \in \mathcal{B}(p, w)$ according to some choice criterion, to be specified.

Consumers’ choice problem

- Preference approach: the tastes of the decision maker are primitives (given) and embodied in her preferences over alternatives. Axioms of rationality imposed on preferences, then examine behavior.
- Revealed Preference approach: individual’s choice behavior first, impose assumptions on choices then reconstruct underlying consistent preferences.
- The two approaches can be reconciled. The first one prevails in courses.

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- Consumer preferences are represented by a binary relation $\succsim$ on elements of the consumption set X,

$$x \succsim y \qquad \textit{x is at least as good as y.}$$

- Define:
  - Strict preference: $x \succ y$ if, and only if, $x \succsim y$ but $y \not\succsim x$;
  - Indifference: $x \sim y$ if, and only if, $x \succsim y$ and $y \succsim x$.

- The binary relation $\succsim$ compares *two* consumption plans at a time.
  - The same is true for strict preference $\succ$ and indifference $\sim$.
- The following axioms determine basic criteria these binary comparisons must adhere to.

- **Axiom 1 (Completeness):** The binary relation $\succsim$ is *complete* if for all $x, y \in X$, we have that

$$x \succsim y \qquad \text{or} \qquad y \succsim x \text{ (or both).}$$

Remark: if $\succsim$ is complete, then $\succsim$ is *reflexive*, i.e., $x \succsim x$ for all $x \in X$.

- **Axiom 2 (Transitivity):** The binary relation $\succsim$ is *transitive* if for all $x, y, z \in X$,

$$\text{if } x \succsim y \text{ and } y \succsim z, \qquad \text{then } x \succsim z.$$

Definition

A *preference relation* is a complete and transitive binary relation.

- Preference relations that satisfy Axiom 1 (completeness) and Axiom 2 (transitivity) are *rational*.
- In this course we will focus on rational preference relations.

- Given the preference relation $\succsim$ and a consumption bundle x, we define the following subsets of X:

1. the set of bundles that are *at least as good as* x :
$\succsim(x) = \{x' \in X : x' \succsim x\}$, i.e. the upper contour set of x;
2. the set of bundles that are *no better than* x : $\precsim(x) = \{x' \in X : x \succsim x'\}$, i.e. the lower contour set of x;
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- **Axiom 3 (Monotonicity):** The preference relation $\succsim$ is *monotone* if for each $x \in X$ and $y \gg x$, then $y \succ x$.

- **Axiom $3'$ (Strong Monotonicity):** The preference relation $\succsim$ is *strongly monotone* if for each $x \in X$ and $y \geq x$ and $y \neq x$, then $y \succ x$.

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- **Axiom $3'$ (Strong Monotonicity):** The preference relation $\succsim$ is *strongly monotone* if for each $x \in X$ and $y \geq x$ and $y \neq x$, then $y \succ x$.

- **Axiom** $3''$ **(Local Non–Satiation):** The preference relation $\succsim$ is *locally non–satiated* if for each $x \in X$ and for each $\varepsilon > 0$, there exists $y \in X$ such that $||y - x|| \leq \epsilon$ and $y \succ x$.

$|| \cdot ||$ is the Euclidean distance, defined as $\left[\sum_{l=1}^{L} (y_l - x_l)^2 \right]^{1/2}$

Preferences

- Monotonicity has implications on how upper contour sets and lower contour sets of $x \in X$...
- Local Non-Satiation implies that the Indifference set of $x \in X$ is not thick!!

- **Axiom 4 (Convexity):** The preference relation is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex, that is take two bundles $y \succsim x$ and $z \succsim x$, then their convex combination $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0,1]$.

- **Axiom** $4'$ **(Strict Convexity):** The preference relation is *strictly convex* if for every $x \in X$, we have that $y \succsim x$ and $z \succsim x$ with $y \neq z$ imply $\alpha y + (1-\alpha)z \succ x$ for any $\alpha \in [0,1]$.

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- **Axiom 5 (Continuity):** The preference relation $\succsim$ is *continuous* if for each sequence of pairs of bundles $\{x^n, y^n\}$,

  that verify $x^n \succsim y^n$ for each n, and

  that are converging $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$,

  we have that $x \succsim y$.

- The preference relation is preserved under the limit.

- An equivalent statement of *continuity* of $\succsim$ is that for each $x \in X$ the upper contour set $\succsim (x)$ and the lower contour set $\precsim (x)$ are closed subsets of X.

- Since $\sim (x) = \succsim (x) \bigcap \precsim (x)$, $\sim (x)$ is also closed if continuity holds. Indeed, the intersection of closed sets is closed.

Definition

The consumption bundle $x^* \in X$ is a *satiation point* of $\succsim$ if $x^* \succsim x$ for all $x \in X$.

If $\succsim$ is locally non–satiated, then $\succsim$ has no satiation point.

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Utility Functions

- Preference relations satisfying Axioms 1- 5 (or their variants) discipline consumer behavior, but are difficult to work with.

- Microeconomic theory has developed a more suitable approach to represent consumer's preferences.

- First, we establish the existence of a (ordinal) function that represents well–behaved preference relations, i.e. the *utility function*.

- Then we move on to study the properties of such function when the consumer must choose her most preferred alternative.

Utility Functions

Definition

An utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if for all $x, x' \in X$,

$$u(x') \geq u(x) \text{ if and only if } x' \succsim x.$$

Lemma 2

Suppose that $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X. Then $\succsim$ is a rational preference relation.

- *Completeness:* For any $x, y \in X$, either $u(x) \geq u(y)$ or $u(x) \leq u(y)$.

  Since u represents $\succsim$, we then have that either $x \succsim y$ or $y \succsim x$, that is $\succsim$ is complete.

- *Transitivity:* Take any $x, y, z \in X$ such that $u(x) \geq u(y)$ and $u(y) \geq u(z)$, then $u(x) \geq u(z)$.

  Since u represents $\succsim$, we have that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$, that is $\succsim$ is transitive. ■

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Utility Functions

- Lemma 2 says that if a binary relation is represented by a utility function, then it is complete and transitive, which qualifies a (rational) preference relation.
- In general, is the converse also true? That is, can every preference relation be represented by a utility function?
- The answer is negative, let me show you why by means of an example.

- Consider a particular preference order: the lexicographic order in $\mathbb{R}^2_+$, that is the binary relation $\succsim_\ell$ such that $x \succsim_\ell y$ is defined by looking at the ordered components of the bundles. That is,

$$(x_1, x_2) \succsim_\ell (y_1, y_2) \text{ if, and only if, } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \geq y_2.$$

- Rank elements by comparing the quantities of each good in turn.

- **We show that $\succsim_\ell$ is a preference order, but it cannot be represented by an utility function.**

- The lexicographic order is a preference order, i.e. $\succsim_\ell$ is **complete** and transitive.

- **Completeness.** Take two bundles $x, y \in \mathbb{R}^2_+$. Focus on the first commodity, it must be that either $x_1 > y_1$ or that $x_1 \leq y_1$.

  - If $x_1 > y_1$, then $x \succsim_\ell y$.

  - If $x_1 = y_1$, it could be one of two possibilities: either $x_2 \geq y_2$ or $y_2 \geq x_2$, which respectively lead to either $x \succsim_\ell y$ or $y \succsim_\ell x$.

  - If $x_1 < y_1$, then $y \succsim_\ell x$.

- The lexicographic order is a preference order, i.e. $\succsim_\ell$ is complete and **transitive**.

- **Transitivity.** Take three bundles $x, y, z \in \mathbb{R}^2_+$ such that $x \succsim_\ell y$ and $y \succsim_\ell z$. By the lexicographic order, it could be that
  - $x_1 > y_1 > z_1$, then $x \succsim_\ell z$ is an immediate implication;
  - $x_1 = y_1$ and $x_2 \geq y_2$ and $y_1 > z_1$, in which case again $x \succsim_\ell z$ is implied;
  - $x_1 = y_1$ and $x_2 \geq y_2$, and $y_1 = z_1$ and $y_2 \geq z_2$, in which case again $x \succsim_\ell z$ is implied.

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- The proof goes by contradiction. Suppose that $u : \mathbb{R}^2_+ \to \mathbb{R}$ represents $\succsim_\ell$.
- Since $(x, 1) \succ_\ell (x, 0)$ for all $x \in \mathbb{R}_+$, then $u(x, 1) > u(x, 0)$ for all $x \in \mathbb{R}_+$
- For each $x \in \mathbb{R}_+$, pick a $q_x \in \mathbb{Q}$, such that $u(x, 1) > q_x > u(x, 0)$.
- If $x' > x$, then $(x', 0) \succ_\ell (x, 1)$, it is also true that $q_{x'} > u(x', 0) > u(x, 1) > q_x$.
- Thus, $q(.)$ is a one-to-one function that associates to each real number a rational number.
- This is impossible!! since the set of rational number is a countable set, while the set of reals is uncountable. ■

Utility Functions

Lexicographic preferences

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- This is impossible!! since the set of rational number is a countable set, while the set of reals is uncountable. $\blacksquare$

- The big issue with the lexicographic order is that it is not continuous. This discontinuity allows for sudden reversals of preferences.

- Continuity is a crucial property for the existence of an utility representation of a preference relation.

Theorem 1

Suppose $\succsim$ is a continuous and rational preference relation on X. Then, there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succsim$.

- We establish the result when $X = \mathbb{R}^L_+$ and $\succsim$ is monotone. For ease of exposition let $L = 2$
- Let $e = (1, 1)$ denote the two-dimensional vector with all elements equal to 1. For each $x \in \mathbb{R}^2_+$, let $A^-(x) = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ and $A^+(x) = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$.
- Since $\succsim$ is assumed to be monotone, $x \succsim \mathbf{0}$, and therefore $\mathbf{0} \in A^-(x)$ for all $x \in \mathbb{R}^2_+$, i.e. $A^-(x)$ is non–empty.
- Monotonicity of $\succsim$ implies that $A^+(x)$ is also non–empty: indeed, given x we can find $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$ which belongs to $A^+(x)$.

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- Monotonicity of $\succsim$ implies that $A^+(x)$ is also non–empty: indeed, given x we can find $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$ which belongs to $A^+(x)$.

- Continuity of $\succsim$ implies the upper contour set and the lower contour set of x are closed. Hence, also $A^-(x)$ and $A^+(x)$ are non-empty and closed for every $x \in \mathbb{R}^2_+$.

- Fix $x \in \mathbb{R}^2_+$. Since $\succsim$ is complete, $\mathbb{R}_+ = A^+(x) \bigcup A^-(x)$.

- Thus, $A^+(x) \bigcap A^-(x) \neq \emptyset$, otherwise $\mathbb{R}_+$ would be the union of two disjoint sets, which is not possible since $\mathbb{R}_+$ is connected.

- Hence, there exists a scalar $\hat{\alpha} \in A^+(x) \bigcap A^-(x)$ such that $\hat{\alpha} e \sim x$.

- Since $\succsim$ is monotone, such scalar is unique. Indeed, by monotonicity $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Let $\hat{\alpha}(x)$ denote the unique scalar satisfying $\hat{\alpha}(x) e \sim x$.

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- Fix $x \in \mathbb{R}^2_+$. Since $\succsim$ is complete, $\mathbb{R}_+ = A^+(x) \bigcup A^-(x)$.
- Thus, $A^+(x) \bigcap A^-(x) \neq \emptyset$, otherwise $\mathbb{R}_+$ would be the union of two disjoint sets, which is not possible since $\mathbb{R}_+$ is connected.
- Hence, there exists a scalar $\hat{\alpha} \in A^+(x) \bigcap A^-(x)$ such that $\hat{\alpha} e \sim x$.
- Since $\succsim$ is monotone, such scalar is unique. Indeed, by monotonicity $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Let $\hat{\alpha}(x)$ denote the unique scalar satisfying $\hat{\alpha}(x)e \sim x$.

- Continuity of $\succsim$ implies the upper contour set and the lower contour set of x are closed. Hence, also $A^-(x)$ and $A^+(x)$ are non-empty and closed for every $x \in \mathbb{R}^2_+$.
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- Let our utility function, $u : X \to \mathbb{R}$, be such that $u(x) = \hat{\alpha}(x)$ for every $x \in X$, with $\hat{\alpha}(x)$ being the unique real number in $A^+(x) \bigcap A^-(x)$.

- We need to check that:
  a.) u represents $\succsim$, that is for all $x, x' \in X$,

$$u(x') \geq u(x) \;\text{ if and only if } x' \succsim x.$$

  b.) u is continuous.

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- *[If]* Consider, $\hat{\alpha}(x)$ and $\hat{\alpha}(x')$, such that $\hat{\alpha}(x) \geq \hat{\alpha}(x')$. By construction, since $\succsim$ are monotone, $\hat{\alpha}(x)e \succsim \hat{\alpha}(x')e$, that implies $x \succsim x'$.
- *[Only if]* Suppose alternatively that $x \succsim x'$, then $\hat{\alpha}(x)e \succsim \hat{\alpha}(x')e$, which implies that $\hat{\alpha}(x) \geq \hat{\alpha}(x')$.

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Everything goes through with an arbitrary finite L.

- The idea behind the proof of Theorem 1 is to construct an utility function that selects the value of α that makes the individual indifferent between the bundles x and αe.

- Continuity and monotonicity of preferences are indispensable here to guarantee the uniqueness of such value, in particular when the consumption set is infinite, as $X = \mathbb{R}^L_+$.

- From now on, we focus on continuous preferences $\succsim$, hence representable by a continuous utility function.

Definition. Let $u : X \to \mathbb{R}$ and denote the image of u by $\mathcal{U}$. Consider a strictly increasing function, $\tau : \mathcal{U} \to \mathbb{R}$, we call the function v which is the composition of τ and u, so that $v(x) = \tau(u(x))$, a *monotone transformation* of u.

Notice that $v : X \to \mathbb{R}$ is itself a function from X to $\mathbb{R}$.

Theorem 2

Let $\succsim$ be a preference relation on X and let $u : X \to \mathbb{R}$ be a utility function that represents $\succsim$. Then $v : X \to \mathbb{R}$ also represents $\succsim$ if, and only if, v is a monotone transformation of u.

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- If $u : X \to \mathbb{R}$ represents $\succsim$, then monotonicity of preferences implies that the utility function u is increasing.

- If $u : X \to \mathbb{R}$ represents $\succsim$, then monotonicity of the preference relation implies that the utility function u is increasing.

- Suppose $u : X \to \mathbb{R}$ represents $\succsim$, convexity of the preference relation implies that u is quasi–concave.

- If $u : X \to \mathbb{R}$ represents $\succsim$, then convexity of $\succsim$ implies that the utility function u is quasi–concave.

  The function $u : X \to \mathbb{R}_+$ is quasi-concave if :

  i.) for every $x \in X$ the set $\{y \in X : u(y) \geq u(x)\}$ is convex,

  ii.) or, equivalently, for every $x, y \in X$, $u(\alpha x + (1-\alpha)y) \geq Min\{u(x), u(y)\}$ for every $\alpha \in [0, 1]$.

- A preference relation $\succsim$ that is *strictly convex* implies that u is *strictly quasi–concave*.

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- Take $L = 2$ and consider an utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ that represents a convex preference relation $\succsim$ in $X = \mathbb{R}^2_+$.

- Assume u is twice continuously differentiable $\mathcal{C}^2$.

- Let $c \in \mathcal{U}$ be an element in the image of u. We construct an *indifference curve*

$$u(x_1, x_2) = c. \tag{1}$$

that is the locus of all pairs $(x_1, x_2) \in \mathbb{R}^2_+$ that yield the same utility level c to the consumer.

Indifference Curve

- Let $X_1 = \{x_1 \in \mathbb{R}_+ : \exists x_2 \in \mathbb{R}_+ \text{ s.t. } u(x_1, x_2) = c\}$. By construction,
  - X_1 is non–empty;
  - for each $x_1 \in X_1$, since u is quasi-concave there exists a *unique* $x_2 \in \mathbb{R}_+$ such that $u(x_1, x_2) = c$.
- Equation (1) then defines a function $f : X_1 \to \mathbb{R}_+$ such that $u(x_1, f(x_1)) = c$ for all $x_1 \in X_1$.

- Since all pairs of bundles (x_1, x_2) which belong to a given indifference curve yield to the consumer the same level of utility, say c. Then, by totally differentiating (1), we derive that

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial u}{\partial x_1}(x_1, x_2)}{\frac{\partial u}{\partial x_2}(x_1, x_2)} \equiv -\mathrm{MRS}_{1,2}(x_1, x_2) \tag{2}$$

The Consumer's Problem

- Assume $\succsim$ is a rational, continuous and locally non-satiated preference relation, and therefore represented by a continuous utility function u (Theorem 1).

- The consumer's problem (henceforth, *UMP*) is then given by

$$\begin{aligned} \max \quad & u(x) \\ s.t. \quad & w - p \cdot x \geq 0 \\ & x \geq 0 \end{aligned}$$

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- Does this problem have a solution?

- For all $p \gg 0$ and $w > 0$, $\mathcal{B}(p, w)$ is closed and bounded.

  - Bounded: if $x \in \mathcal{B}(p, w)$, then $x_i \geq 0$ and $x_i \leq w/p_i$ for each $i \in \{1, \ldots, n\}$.

  - Closed: let $\{x^k\}$ be a converging sequence in $\mathcal{B}(p, w)$. Since $x^k \geq 0$ for all $k \geq 1$, we have that $\lim x^k = x \geq 0$ as well.

    Consider $g_0(x) = w - p \cdot x$, which is a continuous function in x, that is $g_0(x^k)$ converges to $g_0(x)$, notice that $g_0(x^k) \geq 0$ for all k implying that $g_0(x) \geq 0$. Thus, $x \in \mathcal{B}(p, w)$.

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The Consumer's Problem

Existence

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The Utility Maximization Problem - UMP

$$\begin{array}{ll} \max & u(x) \\ s.t. & w - p \cdot x \geq 0 \\ & x \geq 0 \end{array}$$

- The solution to this problem $x(p, w)$ is called the *Walrasian demand correspondence (function)*.
- We call $v(p, w) = u(x(p, w))$ the *indirect utility function*.
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The Consumer's Problem

Theorem 3

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Walrasian demand correspondence has the following properties:

1. $x(p, w)$ is homogeneous of degree zero in (p, w), i.e. $x(\alpha p, \alpha w) = x(p, w)$ for every $\alpha > 0$;
2. $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$;
3. if $\succsim$ is convex, and $u(.)$ is quasi-concave, then $x(p, w)$ is a convex set. If $\succsim$ is strictly convex, and $u(.)$ is strictly quasi-concave, then $x(p, w)$ is a singleton.

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Let us prove all three properties.

i) Follows immediately from the fact that $\mathcal{B}(p, w) = \mathcal{B}(\alpha p, \alpha w)$ for all $\alpha > 0$.

ii) $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$. It follows from LNS.
Assume by contradiction that $p \cdot x < w$ at an $x \in x(p, w)$.
By LNS, there exists an $\epsilon > 0$ small enough and a bundle y in an ϵ-neighborhood of x, $||y - x|| \leq \epsilon$, such that $y \succ x$ and $p \cdot y \leq w$.
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This contradicts that $x \in x(p, w)$.

iii-a) Take $u(.)$ is quasi-concave ($\succsim$ convex) and let $x \in x(p, w)$ and $x' \in x(p, w)$ solve UMP.

We have to show that $\alpha x + (1-\alpha)x' \equiv x'' \in x(p, w)$ for every $\alpha \in [0, 1]$.

Since x and x' solve UMP, it must be $u(x) = u(x')$, denote it u^*.

Since $u(.)$ is quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') \geq u^*$.

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The Walrasian demand correspondence
Proof of Theorem 3

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iii-b) Take $u(.)$ is strictly quasi-concave ($\succsim$ strictly convex) and assume by contradiction that x, x' with $x \neq x'$ are two solutions to UMP, i.e. $x \in x(p, w)$ and $x' \in x(p, w)$.

Consider $\alpha x + (1-\alpha)x' \equiv x''$ for every $\alpha \in (0, 1)$. Since x and x' solve UMP, it must be $u(x) = u(x')$, denote it u^*.

Since $u(.)$ is strictly quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') > u^*$.

Since $\mathcal{B}(p, w)$ is a convex set, again it holds that $x'' \in \mathcal{B}(p, w)$.

Hence, x'' is a budget-feasible bundle which yields utility $u(x'') > u^*$, hence neither x nor x' can solve UMP. Therefore, $x(p, w)$ must contain only one element. ■

The Consumer's Problem

Theorem 3

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Walrasian demand correspondence has the following properties:

i) $x(p, w)$ is homogeneous of degree zero in (p, w);

ii) $x(p, w)$ satisfies Walras' law;

iii) if $\succsim$ is convex, $x(p, w)$ is a convex set. If $\succsim$ is strictly convex, $x(p, w)$ is a singleton.

The Walrasian demand function

- When $\succsim$ is strictly convex, the solution of UPM is the Walrasian demand function, denote it $x^*(p,w)$.

- Let us now focus on $x^*(p,w)$ for some comparative-statics exercises.

- We discuss wealth effects and price effects.

- Fix the price level at $\bar{p}$, and consider $x^*(\bar{p}, w)$ as a function of w, this is the *Engel curve*.

- Consider how the demand function $x^*(\bar{p}, w)$ changes for different values of wealth, the set of all the values $\{x^*(\bar{p}, w) : w > 0\}$ is the wealth expansion path.

- Holding the price level fixed at $\bar{p}$, take $x^*(\bar{p}, w)$ differentiable. We can compute for each commodity k,

$$\frac{\partial x_k^*(\bar{p}, w)}{\partial w}$$

this is the wealth effect on the demand of good k.

- If $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} \geq 0$, good k is a normal good;
- if $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} < 0$, good k is an inferior good.
- How would the wealth expansion path of a normal good look like? and of an inferior good?

- Starting from the Walrasian demand, consider $x^*(p, w)$ as a function of the price vector $p = (p_1, \ldots, p_k, \ldots, p_L)$.

- Consider the demand for commodity k, $x_k^*(p_1, \ldots, p_k, \ldots, p_L, w)$.

  Fix the wealth at $\bar{w}$ and the prices of all commodities except k.

  It is customary to write,
  $p = (p_k, \bar{p}_{-k})$ with $\bar{p}_{-k} = (\bar{p}_1, \ldots, \bar{p}_{k-1}, \bar{p}_{k+1}, \ldots, \bar{p}_L) \in \mathbb{R}^{L-1}_{++}$.

- The set of all values $\{x_k^*(p_k, \bar{p}_{-k}, \bar{w}) : p_k > 0\}$ is the *offer curve* for commodity k.

- Let $x^*(p, w)$ be differentiable. In general,

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_k} < 0,$$

i.e. demand and price of a commodity are inversely related.

- If $\dfrac{\partial x_k^*(p, \bar{w})}{\partial p_k} > 0$ commodity k is a *Giffen good*.

- Think about how would the offer curve for a Giffen good look like. *[Hint: use $X = \mathbb{R}^2_+$]*

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- Think about how would the offer curve for a Giffen good look like. *[Hint: use* $X = \mathbb{R}^2_+$*]*

- We can also evaluate the effect of a change in the price of commodity j, p_j, on the demand for commodity k, $x_k^*(p, \bar{w})$, that is

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j};$$

- if commodity j is a complement for commodity k, the cross-price effect will be negative

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j} < 0;$$

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Consumer Theory

November 17, 2025

Comparative statics and homogeneity of degree zero

- Consider property $i)$ of Theorem 3, for the Walrasian demand function: $x^*(\alpha p, \alpha w) - x^*(p, w) = 0$. For each commodity j

$$x_j^*(\alpha p_1, \ldots, \alpha p_L, \alpha w) - x_j^*(p_1, \ldots, p_L, w) = 0$$

- Differentiate it w.r.t. α and evaluate at $\alpha = 1$. We get the following result.

Homogeneity of degree zero of the Walrasian demand implies that for all p and w,

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p, w)}{\partial p_k} p_k + \frac{\partial x_j^*(p, w)}{\partial w} w = 0 \qquad \text{for } j = 1, \ldots, L \tag{3}$$

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Comparative statics and homogeneity of degree zero

- Take equation (3) and commodity j, divide each addend by $x_j^*(p,w)$, we get
$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} \frac{p_k}{x_j^*(p,w)} + \frac{\partial x_j^*(p,w)}{\partial w} \frac{w}{x_j^*(p,w)} = 0$$

- Recall that $\epsilon_{jk} = \frac{\partial x_j^*(p,w)}{\partial p_k} \frac{p_k}{x_j^*(p,w)}$ is the elasticity of the demand for commodity j to the price of commodity k,

  and $\epsilon_{jw} = \frac{\partial x_j^*(p,w)}{\partial w} \frac{w}{x_j^*(p,w)}$ is the wealth elasticity of the demand for commodity j.

Comparative statics and homogeneity of degree zero

- Then, equation (3)

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} p_k + \frac{\partial x_j^*(p,w)}{\partial w} w = 0$$

rewritten in terms of price and wealth elasticities, yields

$$\sum_{k=1}^{L} \epsilon_{jk} + \epsilon_{jw} = 0 \qquad \text{for } j = 1, \ldots, L$$

- When all prices and wealth change by an equal percentage, this leads to no change in demand of commodity j.

- By Walras law, the Walrasian demand is such that $p \cdot x^*(p,w) = w$, which rewrites as

$$p_1 x_1^*(p,w) + p_2 x_2^*(p,w) + \cdots + p_L x_L^*(p,w) = w. \tag{4}$$

a.) For each commodity j evaluate the differential change in (4) due to a change in $(p_1, ...p_L)$.

By Walras' law, for all p and w, for each commodity j,

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} p_j + x_j^*(p,w) = 0 \qquad \text{for } j = 1, \ldots, L$$

For each commodity, the effect of a change in prices on its expenditure must be zero. Overall, the total expenditure cannot change if prices change.

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Consumer Theory

November 17, 2025

The Walrasian demand and the Weak Axiom

Lemma

The Walrasian demand function $x(p,w)$ satisfies the following property: for every two pairs (p,w), (p',w')

$$\text{if } p \cdot x(p',w') \leq w \text{ and } x(p',w') \neq x(p,w) \qquad \text{then } p' \cdot x(p,w) > w'.$$

This is a property of consistency in the choices of the consumer. Indeed, since both $x(p,w)$ and $x(p',w')$ solve UMP at the respective prices and wealth; if $x(p',w')$ is feasible at (p,w) but is not chosen and the two bundles are different, then it has to be that $x(p,w)$ is not be feasible at (p',w')...

...otherwise, one would expect the consumer to keep preferring $x(p,w)$ over $x(p',w')$ also at price (p',w')... differently, the consumer would have an inconsistent demand behavior!!

This property is the weak axiom of revealed preferences (WARP).

Implications of WARP on the price effects of the Walrasian demand

- A price change alters the relative cost of a commodity w.r.t. the other commodities in the UMP. (Substitution effect).
- Consider a change in prices accompanied by the (specific) change in w that maintains the initial consumption bundle, just affordable at the new prices.
- Start with (p, w) and the consumer optimal choice $x(p, w)$. Consider a price change to $p' \neq p$, and the change in wealth s.t. $w' = p' \cdot x(p, w)$.
- Then, $\Delta w \equiv w' - w = (p' - p) \cdot x(p, w) \to$ this is the Slutsky wealth compensation.
- The price changes accompanied by such wealth compensation are labelled as (Slutsky) compensated price changes.

WARP and the law of demand in UMP

Proposition (WARP)

The Walrasian demand function $x(p,w)$ satisfies WARP if and only if, for any compensated price change from (p,w) to $(p',w') = (p', p' \cdot x(p,w))$, we have

$$(p'-p) \cdot [x(p',w') - x(p,w)] \leq 0$$

with strict inequality whenever $x(p',w') \neq x(p,w)$.

- The Weak Axiom imposes not only a certain consistency on the Walrasian demand, but also a form of the law of demand, in that the change in prices and in the Walrasian demands move in opposite directions... at least for compensated price changes!!

- Read the differential version of Proposition (WARP) - MWG chapter 2, p. 33-34. We shall discuss it soon in class.

The indirect utility function $v(p, w)$

Theorem 4

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the indirect utility function has the following properties:

i) $v(p, w)$ is homogeneous of degree zero in (p, w);

ii) $v(p, w)$ is strictly increasing in w and non-increasing in p;

iii) $v(p, w)$ is quasi-convex, that is the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$;

iv) $v(p, w)$ is continuous in p and w.

Let us prove properties $i) - ii)$.

i) Follows immediately from the fact that $x(p, w)$ is homogeneous of degree zero in (p, w).

   Indeed since $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$, then also $u(x(\alpha p, \alpha w)) = u(x(p, w))$.

ii) We prove that $v(p, w)$ is increasing in w.

Take $w' > w$, then $\mathcal{B}(p, w) \subseteq \mathcal{B}(p, w')$.

In particular, if x^* is the optimal bundle at wealth w, then x^* is feasible when wealth is w'. Hence, $v(p, w') \geq u(x^*) = v(p, w)$.

Since $\succsim$ is LNS, there exists x' that $||x' - x^*|| \leq \epsilon$ such that $x' \succ x^*$ and $x' \in \mathcal{B}(p, w')$ when ϵ is small enough, hence $u(x') > u(x^*)$.

Thus, $v(p, w') \geq u(x') > u(x^*) = v(p, w)$.

Using a similar reasoning, show that $v(p, w)$ is non-increasing in p.

The Expenditure Minimization Problem

The Expenditure Minimization Problem

- The basic idea behind the consumer's problem is to choose a bundle that maximizes utility without violating feasibility.

- There is a second way for the consumer to choose a consumption bundle: pick the least costly bundle that yields him a desired utility level.

- This second form of choice is the one we explore now.

The Expenditure Minimization Problem

- Let $\mathcal{U} = \{u(x) : x \in \mathbb{R}^L_+\}$ denote the set of attainable utility levels.

- For each $\overline{u} \in \mathcal{U}$ and $p \gg 0$, the *expenditure minimization problem (EMP)* is

$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t.} & u(x) \geq \overline{u} \\ & x \geq 0 \end{array}$$

- We denote $h(p, \overline{u})$ the solution to this problem, this is the *Hicksian (or compensated) demand correspondence*.

- The value function $e(p, \overline{u}) = p \cdot h(p, \overline{u})$ is called the *expenditure function*.

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- Existence and uniqueness of a solution to *EMP* are guaranteed by $p \gg 0$ and continuity and strict convexity of $\succsim$.

- These are the same conditions that guarantee existence and uniqueness of a solution to *UMP*.

- In what follows, we assume the existence conditions are satisfied.

Duality

- In neoclassical theory, *UMP* and *EMP* are two mirroring ways to look at the same problem.

- On one hand, in *UMP* the consumer seeks to maximize utility given a fixed wealth/expenditure, namely w.

- On the other hand, in *EMP* the consumer seeks to minimize the expenditure necessary to reach a certain utility level, $\bar{u}$.

- We can formally state this intuition.

Duality: implications for the value functions

Theorem 8

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$. Then,

i) if x^* is optimal in *UMP* when wealth is $w > 0$, then x^* is optimal in *EMP* when the required utility is $u(x^*)$. Moreover, the minimized expenditure level in *EMP* is exactly w, that is $p \cdot x^* = w$;

ii) if x^* is optimal in *EMP* at utility $\overline{u} > u(0)$, then x^* is optimal in *UMP* when wealth is equal to $p \cdot x^*$. Moreover, the maximized level of utility in *UMP* is exactly $\overline{u}$, that is $u(x^*) = \overline{u}$.

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2. if x^* is optimal in *EMP* at utility $\bar{u} > u(0)$, then x^* is optimal in *UMP* when wealth is equal to $p \cdot x^*$. Moreover, the maximized level of utility in *UMP* is exactly $\bar{u}$, that is $u(x^*) = \bar{u}$.

We prove i).

i) By contradiction, assume x^* solves UMP, but is not optimal in *EMP* when the required utility is $u(x^*)$.

Then, there must exist an x' such that $p \cdot x' < p \cdot x^*$ and $u(x') \geq u(x^*)$. Since x^* solves UMP, $p \cdot x^* \leq w$.

By LNS of $\succsim$, we can find x'' in an ϵ-ball around x', i.e. $||x'' - x'|| \leq \epsilon$, that satisfies $p \cdot x'' < w$ and $x'' \succ x' \ \leftrightarrow u(x'') > u(x')$.

This implies that $x'' \in \mathcal{B}(p, w)$ and that x^* is not optimal in UMP. A contradiction.

Hence, x^* is optimal in *EMP* and the minimized expenditure is $p \cdot x^*$. Since x^* solves UMP, it satisfies Walras' law, i.e. $p \cdot x^* = w$.

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Let us prove part *ii*) of Theorem 8.

ii) Observe that since $\overline{u} \geq u(0)$, then $x^* \neq 0$ and $p \cdot x^* > 0$.

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Take $x'' = \alpha x'$ with $\alpha \in (0, 1)$. By continuity of $u(.)$, when α is close to 1, $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$.

This implies that x'' is preferred to x^* in *EMP*, since it guarantees the desired utility at lower expenditure. A contradiction.

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Expenditure Minimization Problem: Summary

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$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t.} & u(x) \geq \overline{u} \\ & x \geq 0 \end{array}$$

- The solution to EMP is the *Hicksian (or compensated) demand correspondence*, $h(p, \overline{u})$;

- The value function of EMP is the *expenditure function*, $e(p, \overline{u}) = p \cdot h(p, \overline{u})$.

- EMP is dual to UMP.

Properties of the Hicksian demand $h(p, \overline{u})$

Parallel to what we did for *UMP*, let us now examine the properties of the Hicksian demand and of the expenditure function.

Theorem 5

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Hicksian demand correspondence has the following properties:

i) it is homogeneous of degree zero in prices, i.e. $h(\alpha p, \overline{u}) = h(p, \overline{u}) \; \forall \; \alpha > 0$;

ii) it satisfies no excess utility, i.e. $u(x) = \overline{u}$ for every $x \in h(p, \overline{u})$;

iii) if $\succsim$ is convex, then $h(p, \overline{u})$ is a convex set. If $\succsim$ is strictly convex, then $h(p, \overline{u})$ is single-valued.

Let us prove all three properties.

i) Immediate: the optimal bundle x that minimizes $p \cdot x$ also minimizes $\alpha p \cdot x$ for every $\alpha > 0$, subject to the same constraint $u(x) \geq \bar{u}$.

ii) $h(p, w)$ satisfies no excess utility, i.e. $u(x) = \overline{u}$ for every $x \in h(p, \overline{u})$.

Follows from continuity of $u(.)$. Assume by contradiction that $x \in h(p, \overline{u})$ is such that $u(x) > \overline{u}$.

Consider a bundle $x' = \beta x$, with $\beta \in (0, 1)$.

By continuity of $u(.)$, when β is close enough to 1, $u(x') \geq \overline{u}$ and $p \cdot x' < p \cdot x$.

Then, $x \notin h(p, \overline{u})$, a contradiction.

Hence, $u(x)$ must be equal to $\overline{u}$ for every $x \in h(p, \overline{u})$.

The Hicksian demand correspondence

Proof of Theorem 5 - iii.)

iii-a) Take $\succsim$ convex and let x, x' be two solutions to EMP, i.e. $x \in h(p, \overline{u})$ and $x' \in h(p, \overline{u})$.

We have to show that $\alpha x + (1-\alpha)x' \equiv x'' \in h(p, \overline{u})$ for every $\alpha \in [0, 1]$.

Since x and x' solve EMP, it must be that $p \cdot x = p \cdot x' = e^*$.

Hence, $p(\alpha x + (1-\alpha)x') = \alpha p \cdot x + (1-\alpha)p \cdot x' = e^*$, that is any convex combination of solutions of EMP is itself expenditure minimizing.

In addition, we also know that $u(x) = u(x') = \overline{u}$ by *ii.*). Since $u(.)$ is quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') \geq \overline{u}$.

Hence, x'' is an expenditure minimizing bundle which yields utility $u(x'')$ not lower than $\overline{u}$, therefore is must be $x'' \in h(p, \overline{u})$, too. $h(p, \overline{u})$ is a convex set.

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Hence, x'' is an expenditure minimizing bundle which yields utility $u(x'')$ not lower than $\overline{u}$, therefore is must be $x'' \in h(p, \overline{u})$, too. $h(p, \overline{u})$ is a convex set.

iii-b) Take $\succsim$ strictly convex, i.e. $u(.)$ strictly quasi-concave, then $h(p, \overline{u})$ contains a single element.

Prove it!!

Properties of the Expenditure Function $e(p, \overline{u})$

Theorem 6

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the expenditure function has the following properties:

1. it is homogeneous of degree one in prices, i.e. $e(\alpha p, \overline{u}) = \alpha e(p, \overline{u}) \; \forall \; \alpha > 0$;
2. it is strictly increasing in $\overline{u}$ and non-decreasing in p_l for every $l = 1, \ldots, L$;
3. it is concave in p;
4. continuous in p and $\overline{u}$.

Let us prove properties $i) - iii)$.

i) Follows immediately from the fact that $h(p, \overline{u})$ is homogeneous of degree zero in $(p, \overline{u})$. Indeed since $h(\alpha p, \overline{u}) = h(p, \overline{u})$ for all $\alpha \in [0, 1]$, then also $\alpha p \cdot h(\alpha p, \overline{u}) = \alpha p \cdot h(p, \overline{u})$.

The Expenditure Function $e(p, \overline{u})$

Proof of Theorem 6 - ii)

ii-a) We prove that $e(p, \overline{u})$ is strictly increasing in $\overline{u}$.

Suppose, by contradiction, that $e(p, \overline{u})$ is not strictly increasing in $\overline{u}$, and let x' and x'' denote optimal consumption bundles for utility levels u' and u'', respectively, with $u'' > u'$ and $p \cdot x' \geq p \cdot x'' > 0$.

Consider a bundle $\tilde{x} = \beta x''$, where $\beta \in (0, 1)$.

By continuity of $u(\cdot)$, there exists a β close enough to 1 such that $u(\tilde{x}) > u'$ and $p \cdot x' > p \cdot \tilde{x}$, a contradiction.

ii-b) To show that $e(p, \overline{u})$ is non-decreasing in p_l, consider the price vectors p'' and p' such that $p''_l \geq p'_l$ for commodity l, and $p''_k = p'_k$ for all commodities $k \neq l$.

Let x'' be the solution to the *EMP* for prices p''.

Then, $e(p'', \overline{u}) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', \overline{u})$, where the latter inequality follows from the definition of $e(p', \overline{u})$.

The Expenditure Function $e(p, \overline{u})$

Proof of Theorem 6 - iii)

iii) To show that $e(p, \overline{u})$ is concave in p, we need to prove that

$$e(\alpha p + (1-\alpha)p', \overline{u}) \geq \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$$

for every p, p' and for every $\alpha \in [0, 1]$.

Denote $p'' \equiv \alpha p + (1-\alpha)p'$ and let $x'' \in h(\alpha p + (1-\alpha)p', \overline{u})$ be a solution to EMP at price p''.

iii) To show that $e(p, \overline{u})$ is concave in p, we need to prove that

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for every p, p' and for every $\alpha \in [0, 1]$.

Denote $p'' \equiv \alpha p + (1-\alpha)p'$ and let $x'' \in h(\alpha p + (1-\alpha)p', \overline{u})$ be a solution to EMP at price p''.

Then,

$$e(p'', \overline{u}) = p'' \cdot x'' = (\alpha p + (1-\alpha)p') \cdot x'' =$$

$$\alpha p \cdot x'' + (1-\alpha)p' \cdot x'' \geq \alpha p \cdot h(p, \overline{u}) + (1-\alpha)p' \cdot h(p', \overline{u})$$

indeed, x'' is a sub-optimal choice when the prices are either p or p'.

Since $\alpha p \cdot h(p, \overline{u}) + (1-\alpha)p' \cdot h(p', \overline{u}) = \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$, the chain of inequalities above expresses the concavity of the expenditure function.

The Walrasian Demand Correspondence. The Hicksian Demand Correspondence. The law of demand.

November XX, 2025

WARP and the law of demand in UMP

Proposition (WARP)

The Walrasian demand function $x(p,w)$ satisfies WARP if and only if, for any compensated price change from (p,w) to $(p',w') = (p', p' \cdot x(p,w))$, we have

$$(p' - p) \cdot [x(p',w') - x(p,w)] \leq 0$$

with strict inequality whenever $x(p',w') \neq x(p,w)$.

- The Weak Axiom imposes a consistency requirement on the Walrasian demand, and implies a form of the law of demand, in that the change in prices and in Walrasian demands move in opposite directions for every compensated price change!!

Compensated change in prices and Walrasian demand

- Consider commodity l and the effect on $x_l(p, w)$ of a compensated change in the price of p_l, only.

- $\Delta p = p' - p = (0, \ldots, \Delta p_l, \ldots, 0)$. We want to measure Δx_l, Proposition WARP implies that if $\Delta p_l > 0$ then $\Delta x_l < 0$.

- We cannot say much about the effect of a price change that is not compensated!

- Consider the differential version of Proposition WARP

$$dp \cdot dx \leq 0$$

for a compensated change in wealth induced by a change in the price vector.

Substitution effect and Walrasian demand

- In $dp \cdot dx \leq 0$, dx measures the total variation of the (array of) Walrasian demand $x(p', w' = p' \cdot x(p,w))$ induced by the change in price and the compensation in wealth.

$$dx = D_p x(p,w)dp + D_w x(p,w)dw$$

with $D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \dots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$

and $D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \dots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix}$

Substitution effect and Walrasian demand

- Since we deal with compensated price changes, $dw = dp \cdot x(p, w)$. Hence,

$$dx = D_p x(p, w) dp + D_w x(p, w)[dp \cdot x(p, w)]$$

or

$$dx = [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp.$$

Finally,

$$dp \cdot dx = dp \cdot [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp \leq 0$$

Slutsky matrix and substitution effects

$$[D_p x(p, w) + D_w x(p, w) x(p, w)^T] \equiv S(p, w)$$

is an $(L \text{ x } L)$ matrix, called Slutsky matrix, $S(p, w)$, with generic element of row l and column k equal to

$$s_{lk}(p, w) = [\frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)]$$

which is called the substitution effect.

Slutsky matrix and substitution effects

$$s_{lk}(p,w) = [\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)]$$

The substitution effects captures the (differential) change in demand of good l due to a differential change in the price of good k, when wealth is compensated so that the consumer can just afford his original bundle ... hence induced by change in relative prices only.

$\frac{\partial x_l(p,w)}{\partial p_k} dp_k$ measures change in demand of good l if w is unchanged;

$x_k(p,w)dp_k$ measures the compensated change in wealth;

$\frac{\partial x_l(p,w)}{\partial w}[x_k(p,w)dp_k]$ measures the change in demand of good l due to the compensated change in wealth.

Slutsky matrix and substitution effects

- To summarize, $dp \cdot dx \leq 0$ is equivalent to

$$dp \cdot [D_p x(p,w) + D_w x(p,w) x(p,w)^T] dp \leq 0$$

- Since the Walrasian demands satisfy the weak axiom, the Slutsky matrix is negative semi-definite for every (p,w).

- Negative semi-definiteness of $S(p,w)$ implies that $s_{ll}(p,w) \leq 0$ for every $l = 1, 2, ..., L$, own substitution effects are non-positive.

- However, we know that $\frac{\partial x_l(p,w)}{\partial p_l} > 0$ (for Giffen goods), hence for $s_{ll}(p,w) \leq 0$ it has to be $\frac{\partial x_l(p,w)}{\partial w} < 0$. That is, a good can be a Giffen good at some (p,w) only if it is inferior.

The Hicksian demand and the compensated law of demand

Theorem 7

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $h(p, \overline{u})$ uniquely identifies the optimal bundle for all $p \gg 0$. Then, the Hicksian demand function satisfies the compensated law of demand: for all p, p'

$$(p' - p) \cdot \big(h(p', \overline{u}) - h(p, \overline{u})\big) \leq 0$$

- By definition, $h(p, \overline{u})$ solves the expenditure minimization problem at price $p \gg 0$. Hence,

$$\begin{aligned} p' \cdot h(p', \overline{u}) &\leq p' \cdot h(p, \overline{u}) \\ p \cdot h(p, \overline{u}) &\leq p \cdot h(p', \overline{u}) \end{aligned}$$

- Rearranging the two inequalities, we get

$$\begin{aligned} p' \cdot \left[h(p', \overline{u}) - h(p, \overline{u})\right] &\leq 0 \\ -p \cdot \left[h(p', \overline{u}) - h(p, \overline{u})\right] &\leq 0 \end{aligned}$$

summing we get the result.■

- The result we just proved implies that, differently from the Walrasian demand, the change of the Hicksian demand is *always* inverse with respect to any change in prices.

- The inverse relationship holds for each commodity, i.e.

$$(p' - p) \cdot \big(h(p', \overline{u}) - h(p, \overline{u})\big) \leq 0,$$

is a compact way to express that

$$(p'_k - p_k) \cdot \big(h_k(p', \overline{u}) - h_k(p, \overline{u})\big) \leq 0 \qquad \text{for } k = 1, 2, \dots, L.$$

Duality: implications for the value functions

- We have formally shown that EMP is the dual problem of UMP and viceversa.

- More precisely, by Theorem 8, if $u(.)$ is a continuous utility function representing LNS $\succsim$ on $X = \mathbb{R}^L_+$ and if $p \gg 0$,

a) if x^* is optimal in *UMP* at $w > 0$, then x^* is optimal in *EMP* at $u(x^*)$. Moreover, the expenditure function of such *EMP* is exactly equal to w, i.e. $p \cdot x^* = w$;

b) if x^* is optimal in *EMP* at $\overline{u} > u(0)$, then x^* is optimal in *UMP* at wealth equal to $p \cdot x^*$. Moreover, the indirect utility of such *UMP* is exactly equal to $\overline{u}$, i.e. $u(x^*) = \overline{u}$.

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Duality: expenditure function

Theorem 8 supports the following reasoning.

Let $x(p, w)$ be a solution to *UMP* given $p \gg 0$ and $w > 0$, so that

- $p \cdot x(p, w) = w$ (by Walras' law),
- $u(x(p, w)) = v(p, w) \geq \bar{u}$.

Then,

$$e(p, v(p, w)) = p \cdot x(p, w) = w \tag{5}$$

for all $p \gg 0$ and $w > 0$.

An application of duality

Equation (5) states that $e(p, v(p, w)) = w$ for all $p \gg 0$ and $w > 0$.

For example, take $e(p, \bar{u}) = p_1^{\alpha} p_2^{\beta}\ \exp(\bar{u})$. By duality we know that $\bar{u} = v(p, w)$ and that $w = e(p, \bar{u})$. Hence,

$$e(p, v(p, w)) = p_1^{\alpha} p_2^{\beta}\ \exp(v(p, w)) = w$$

Simple computation yields,

$$\exp(v(p, w)) = \frac{w}{p_1^{\alpha} p_2^{\beta}}$$

$$v(p, w) = ln\left(\frac{w}{p_1^{\alpha} p_2^{\beta}}\right) = ln(w) - \alpha ln(p_1) - \beta ln(p_2).$$

Duality: indirect utility

If $h(p, \overline{u})$ is a solution to *EMP* given $p \gg 0$ and $\overline{u} > u(0)$, so that

- $u(h(p, \overline{u})) = \overline{u}$ (no-excess utility),
- $p \cdot h(p, \overline{u}) = e(p, \overline{u}) = w$.

Then,

$$v(p, e(p, \overline{u})) = u(h(p, \overline{u})) = \overline{u} \tag{6}$$

for all $p \gg 0$ and $\overline{u} > u(0)$.

Fix the price vector $p \gg 0$, equations (6) and (5) imply that the indirect utility function and expenditure function are the inverse of one another.

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Fix the price vector $p \gg 0$, equations (6) and (5) imply that the indirect utility function and expenditure function are the inverse of one another.

Duality: implications on demand correspondences

Theorem 8 has known implications on the Walrasian and Hicksian demand correspondences for every commodity. For all $p \gg 0$ and $\overline{u} > u(0)$,

$$x_l(p,w) = h_l(p, v(p,w)) \text{ for each commodity } l = 1, \dots, L$$

$$h_l(p,\overline{u}) = x_l(p, e(p,\overline{u})) \text{ for each commodity } l = 1, \dots, L.$$

Duality: implications on demand correspondences

Take

$$h_l(p, \overline{u}) = x_l(p, e(p, \overline{u}))$$

for some arbitrary commodity l and consider a change in prices. (see note)

The Hicksian demand $h(p, \overline{u})$ measures the demand that would emerge if we adjust wealth so maintain the consumer at the same level of utility.

This type of compensation is the Hicksian wealth compensation, and explains why $h(p, \overline{u})$ is the compensated demand correspondence.

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This type of compensation is the Hicksian wealth compensation, and explains why $h(p, \overline{u})$ is the compensated demand correspondence.

The relationship between Hicksian and Walrasian demands, indirect utility and expenditure functions.

More implications of duality

- Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$.

- Assume also that $\succsim$ is strictly-convex, which implies strictly quasi-concave utility function, hence $x(p, w)$ and $h(p, \overline{u})$ identify a unique optimal bundle for UMP and EMP, respectively.

- We start by examining the relationship between the expenditure function and the Hicksian demand.

Hicksian Demand and Expenditure Function

Shepard's Lemma: the relationship between $e(p, \overline{u})$ and $h(p, \overline{u})$

Hicksian Demand and Expenditure functions

Shepard's Lemma

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$, and that $p \gg 0$.
If $e(p, \overline{u})$ is differentiable in p then, for all p and $\overline{u}$, the Hicksian demand is the derivative of the expenditure function with respect to prices, i.e.

$$\frac{\partial e(p, \overline{u})}{\partial p_k} = h_k(p, \overline{u}) \qquad \text{for } \; k = 1, \cdots, L.$$

- The proof is an implication of the Envelope Theorem.
- In EMP, prices are parameters of the min problem.
- The Envelope Theorem tells us that, in an optimization problem, when measuring the first-order effects of a change in the parameters of the problem on the value function, we can disregard any change in the maximizer (minimizer), and only consider the direct effects.

- Indeed,

$$e(p,\overline{u}) = p \cdot h(p,\overline{u}) = p_1 \cdot h_1(p,\overline{u}) + \cdots + p_L \cdot h_L(p,\overline{u})$$

- Consider commodity k and differentiate $e(p,\overline{u})$ w.r.t. p_k. By the chain rule

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}) + p_1 \frac{\partial h_1(p,\overline{u})}{\partial p_k} + \cdots + p_L \frac{\partial h_L(p,\overline{u})}{\partial p_k}$$

which can be rewritten as

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}) + \sum_{j=1}^{J} p_j \frac{\partial h_j(p,\overline{u})}{\partial p_k}$$

- Indeed,

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Since $h(p,\overline{u})$ is optimal, a change in prices has no first-order effect on demand, i.e. $\frac{\partial h_j(p,\overline{u})}{\partial p_k} = 0$, hence on expenditure, and

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}).$$

When applied to the EMP, the direct effect of a change in p_k on the minimal expenditure, measures the variation of the expenditure $e(p,\overline{u})$ at fixed demand $h(p,\overline{u})$.

Roy's Identity: the relationship between $v(p, w)$ and $x(p, w)$

- Let $u^* = v(p^*, w^*)$. By duality, $v(p, e(p, u^*)) = u^*$ for any p.
- Differentiate with respect to p_j and evaluate at $p = p^*$, we get

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} \frac{\partial e(p^*, u^*)}{\partial p_j} = 0.$$

- Shepard's lemma implies that $\frac{\partial e(p^*, u^*)}{\partial p_j} = h_j(p^*, u^*)$, substituting we get

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} h_j(p^*, u^*) = 0.$$

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- Since $w^* = e(p^*, u^*)$ and $h_j(p^*, u^*) = x_j(p^*, w^*)$, we can write

$$\frac{\partial v(p^*, w^*)}{\partial p_j} + \frac{\partial v(p^*, w^*)}{\partial w} x_j(p^*, w^*) = 0.$$

which gives

$$x_j(p^*, w^*) = -\frac{\dfrac{\partial v(p^*, w^*)}{\partial p_j}}{\dfrac{\partial v(p^*, w^*)}{\partial w}}.$$

Roy's Identity

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$.
Suppose that $v(p,w)$ is differentiable at $(p^*, w^*) \gg 0$. Then, for every $j = 1, \ldots, L$

$$x_j(p^*, w^*) = -\frac{\dfrac{\partial v(p^*, w^*)}{\partial p_j}}{\dfrac{\partial v(p^*, w^*)}{\partial w}}$$

Walrasian Demand and Indirect Utility functions

- Roy's identity is the analog of Shepard's lemma for the Walrasian demand function.

- When deriving the Walrasian demand from the indirect utility, we have to normalize the price derivative of the indirect utility by the derivative of $v(p,w)$ w.r.t. wealth;

$$x_j(p^*,w^*) = -\frac{\frac{\partial v(p^*,w^*)}{\partial p_j}}{\frac{\partial v(p^*,w^*)}{\partial w}}.$$

- Indeed, utility is an ordinal concept, so is the Walrasian demand, which is then sensitive to the underlying $u(.)$.

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Slutsky Equation: the relationship between Hicksian and Walrasian demands

Hicksian and Walrasian demand functions

- Fix $\bar{u} = v(\bar{p}, w)$ for some $\bar{p} \gg 0$ and $w > 0$.
- By duality $w = e(\bar{p}, \bar{u})$.
- Duality also implies that for all p and u and for each commodity $l = 1, \ldots, L$
$$h_l(p, u) = x_l(p, e(p, u)) \tag{7}$$
- Differentiate both sides of (7) with respect to p_k and evaluate it at $(\bar{p}, \bar{u})$ to get
$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}. \tag{8}$$

Hicksian and Walrasian demand functions

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial w}\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} \qquad (8)$$

- By Shepards' Lemma, $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$, hence

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- By duality $e(\bar{p},\bar{u}) = w$ and $h_k(\bar{p},\bar{u}) = x_k(\bar{p},e(\bar{p},\bar{u})) = x_k(\bar{p},w)$, thus $(8')$ can be rewritten as:

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},w)}{\partial p_k} + \frac{\partial x_l(\bar{p},w)}{\partial w}x_k(\bar{p},w).$$

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Hicksian and Walrasian demand functions

The Slutsky Equation

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$. Then, for all (p, w) and $u = v(p, w)$ we have

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \qquad \text{for all } l, k.$$

The Slutsky Equation

- Let $l = k$, the Slutsky equation tells us that

$$\frac{\partial h_l(p,u)}{\partial p_l} = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} x_l(p,w).$$

- For a given commodity, the Slutsky equation relates the slopes of the Hicksian and of the Walrasian demands.
- If commodity l is normal, the Hicksian demand is steeper (more rigid) than the Walrasian demand.
- Indeed, if the price of commodity l increases and its demand falls, the consumer's expenditure increases to guarantee the same level of utility. If such wealth compensation is absent, as in the Walrasian demand, the fall of the demand for commodity l is more pronounced.

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The Slutsky Substitution Matrix

- The matrix that collects all the cross-price derivatives of the Hicksian demands for each commodity l, i.e. $\frac{\partial h_l(p,u)}{\partial p_k}$ for each k, l, is indeed the *Slutsky substitution matrix*, $S(p, w)$.

- Since $S(p, w)$ is obtained by taking the price derivative of the Hicksian demand for each commodity, when demand is generated from EMP, the matrix $S(p, w)$ inherits some properties of the Hicksian demand and of the expenditure function.

The Slutsky Substitution Matrix

Specifically,

- $S(p,w)$ is negative semidefinite [because of Shepard's Lemma and the concavity of $e(p,\overline{u})$];
- $S(p,w)$ is symmetric [i.e. the compensated cross-price derivatives of any two commodities, l and k, are equal, i.e. $\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_l}$];
- $S(p,w)$ is such that $S(p,w) \cdot p = 0$ [by homogeneity of degree zero of $h(p,\overline{u})$].

The property that $S(p,w) \cdot p = 0$, together with the compensated law of demand imply that every commodity has at least one substitute, i.e. since for commodity k, $\frac{\partial h_k(p,u)}{\partial p_k} \leq 0$ there must exist a commodity, say j, such that $\frac{\partial h_j(p,u)}{\partial p_k} \geq 0$.

The Slutsky Equation

The Slutsky equation can be rewritten as follows:

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{TE} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{SE} \underbrace{- \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{IE}$$

The total effect (TE) of a change in price on consumer's demand can be decomposed into two effects: the substitution effect (SE) and the income effect (IE).

SE gives a measure of the effect that a change in price induces in the consumers' demand when wealth is adjusted so to keep the consumer at the same utility level.

IE measures the effect of the same change on the purchasing power of the consumer hence on its Walrasian demand.

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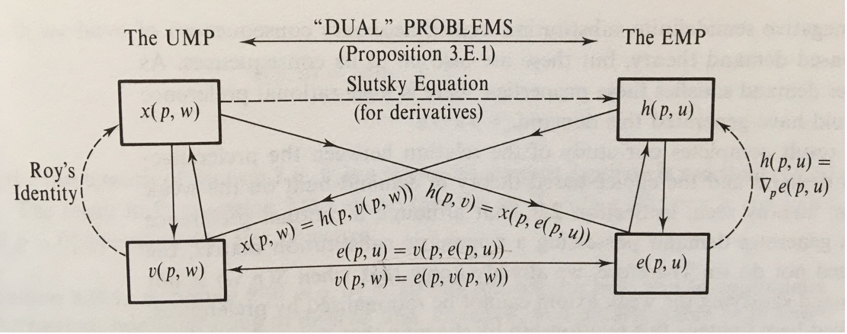

The UMP
"DUAL" PROBLEMS
(Proposition 3.E.1)
The EMP
Slutsky Equation
(for derivatives)
$x(p, w)$
$h(p, u)$
Roy's Identity
$h(p, u) = \nabla_p e(p, u)$
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