

Advanced Microeconomics 2025-26

Theory of Production

a.a. 2025-2026

- We studied the demand side of the economy...
- ... how by disciplining preference relations using axioms we are led to an equivalent representation through the utility functions, and from there to the consumer's demand, as the result of an optimal choice taken over a set of feasible (affordable) alternatives.
- Now, we look at the supply side of the economy.
- The economic agent of interest now is the *firm*.

- Assume the firm is atomistic and non-strategic.
- No specific assumption is made on the market structure in which the firm operates at this stage.
- Being atomistic and non-strategic, it takes the market prices as given, for the production good as well as for the inputs.

- Suppose there are m goods in the economy.
- A *production plan* is a vector $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, in which
 - $y_i > 0$ means good i is an *output*, and y_i is the amount produced;
 - $y_i < 0$ means good i is an *input*, and $-y_i$ is the amount used as input.
- A production plan is feasible if it belongs to the *production set* Y , that is $y \in Y$. Y is exogenously given.

The Production Function

Example with a single output

- When we want to represent a firm that produces a single output, the production set Y is described by means of the *production function*, which represents the technology that uses inputs to produce the output.
- Then, a production plan is $y = (q, z_1, \dots, z_{m-1}) \in \mathbb{R}^m$ in which q is the output, and $(z_1, \dots, z_{m-1})$ represent the amount of goods that can only be used as inputs.
- $f : \mathbb{R}_+^{m-1} \rightarrow \mathbb{R}_+$ is the *production function*, which yields the maximum amount of output that can be obtained combining the inputs $(z_1, \dots, z_{m-1})$ according to the technology $f(\cdot)$
- The production set is $Y = \{y \in \mathbb{R}^m : q - f(z_1, \dots, z_{m-1}) \leq 0\}$.

The Production Function

Marginal Rate of Technical Substitution

- Let the production function f be differentiable. Fix the output level at $\bar{q} = f(\bar{z})$ and consider two inputs l, k .
- The **marginal rate of technical substitution of input l for input k** at $\bar{z}$ is

$$MRTS_{lk}(\bar{z}) = -\frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z})}{\partial z_k}}$$

- $MRTS_{lk}(\bar{z})$ measures by how much the use of input k has to increase, if we decrease the use of input l by one marginal unit, to keep the level of output at $\bar{q}$.
- $MRTS_{lk}(\bar{z})$ measures the slope of the production function at point $\bar{z}$.

Properties of the production set $Y = \{y \in \mathbb{R}^m : q - f(z_1, \dots, z_{m-1}) \leq 0\}$.

- i) Y is **non – empty**.
- ii) Y is **closed**.
- iii) **Possibility of inaction**: $0 \in Y$.
 - The firm can shut down completely. This is possible if there are no sunk costs, i.e., no commitments to the use of some inputs for production.

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iv) **No Free Lunch**: there is no $y > 0$ in Y

- In other words, there is no feasible production plan where some goods are produced with no inputs.

$$Y \cap \mathbb{R}_+^m \subseteq \{0\}$$

v) **Free Disposal**: if $y \in Y$ and $y' \leq y$, then $y' \in Y$

A production plan $y' \leq y$ implies that we can produce lower or equal quantities of outputs as in y with at least as much of the inputs in y .

$$Y - \mathbb{R}_+^m \subseteq Y$$

vi) **Irreversibility**: if $y \in Y$ and $y \neq 0$, then $-y \notin Y$

- This implies that switching the roles of inputs and outputs in a production plan is never feasible.

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vii) Non-Increasing Returns to Scale:

for every $y \in Y$ and $\alpha \in [0, 1]$, the plan $\alpha y \in Y$

Every production plan can be scaled down up to inaction/complete shut down. In other words, $\alpha Y \subseteq Y$ with $\alpha \in [0, 1]$.

viii) Non-Decreasing Returns to Scale:

for every $y \in Y$ and $\alpha > 1$, the plan $\alpha y \in Y$

Every production plan can be scaled up. That is, $\alpha Y \subseteq Y$ for all $\alpha > 1$.

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ix) Constant Returns to Scale:

for every $y \in Y$ and $\alpha \geq 0$, the plan $\alpha y \in Y$

This combines non-increasing and non-decreasing returns to scale. That is, $\alpha Y \subseteq Y$ for all $\alpha \geq 0$

When the returns to scale are constant, the production set is a cone.

x) **Convexity**: Y is convex.

- The production set is convex if for every pair of plans $y, y' \in Y$, $\alpha y + (1 - \alpha)y' \in Y$ for every $\alpha \in [0, 1]$.
- Notice that if inaction is possible, convexity implies non-increasing returns to scale.
Indeed, if Y is convex, then $\alpha y = \alpha y + (1 - \alpha)0 \in Y$ for all $\alpha \in [0, 1]$ and for every $y \in Y$.

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- ① We now show that for the single-output case, Y is convex if and only if $f(z)$ is a concave production function.

xi) **Additivity**: for every $y \in Y$ and $y' \in Y$, the plan $y + y' \in Y$.

- In other words, $Y + Y \subseteq Y$
- It implies that if y is feasible, then replicating it k times is also feasible, i.e., if $y \in Y$ then $ky \in Y$ with $k > 0$.
- We can hence consider an arbitrary number of firms who can possibly become active in the market we are considering. It is a weaker version of non-decreasing returns to scale, it is called *free entry*.

Properties of the Production Set

- Convexity + Constant Returns to Scale imply that Y is a convex cone.

The production set is a **convex cone** if for every pair of plans $y, y' \in Y$ and constants $\alpha \geq 0$ and $\beta \geq 0$, then

$$\alpha y + \beta y' \in Y$$

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Let us now show that

Proposition (A+NIRS)

The production set is additive and satisfies non-increasing returns to scale if and only if it is a convex cone.

- By definition, if Y is a convex cone, then additivity and non-increasing returns to scale are implied.
- We just need to prove that if Y satisfies additivity and non-increasing return to scale, then it is a convex cone, that is
for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

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for every $y, y' \in Y$, for every $\alpha \geq 0$ and $\beta \geq 0$,

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- Assume Y satisfies additivity and non-increasing return to scale.
- Pick arbitrary $\alpha \geq 0$ and $\beta \geq 0$, and let k be any integer such that $k > \max\{\alpha, \beta\}$. By additivity, both $ky \in Y$ and $ky' \in Y$ hold.
- By construction, $\frac{\alpha}{k} < 1$, and $\alpha y = \left(\frac{\alpha}{k}\right)ky \in Y$ by non-increasing returns to scale.
- Similarly, $\frac{\beta}{k} < 1$ and $\beta y' = \left(\frac{\beta}{k}\right)ky' \in Y$.
- Again, by additivity $\alpha y + \beta y' \in Y$. ■

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One way to think about the assumption of convexity of the production set is that it results as an implication of

- additivity, i.e. the possibility to replicate feasible production plans in bigger scale without creating negative interferences to the production activity, and
- non-increasing returns to scale, i.e. the possibility to always scale down any input-output combination.

- Examine the market behavior of an individual firm.
- The firm takes as given the market price of each good, $(p_1, \dots, p_m)$.
- We consider that the objective of the firm is to maximize profits.

The Profit Maximization Problem (PMP)

- Let $p = (p, w_1, \dots, w_{m-1}) \gg 0$ be the vector of prices of the unique output good p and of the $(m - 1)$ inputs w_i with $i = 1, \dots, m - 1$.
- The profit maximization problem (PMP) of the firm is then

$$\begin{aligned} \max_{q, z \geq 0} \quad & p \cdot q - w \cdot z \\ \text{s.t.} \quad & q - f(z) \leq 0 \end{aligned}$$

- In this formulation, the firm chooses quantities of factor inputs to maximise the difference between total revenues and costs.

Lemma 1

If Y is closed, bounded and satisfies free-disposal, the profit maximization problem (*PMP*) has a solution.

PMP: supply correspondence and profit function

- Given $p = (p, w_1, \dots, w_{m-1}) \gg 0$, the profit maximization problem (PMP) of the firm is

$$\begin{aligned} \max_{q,z} \quad & p \cdot q - w \cdot z \\ \text{s.t.} \quad & q - f(z) \leq 0 \\ & z \geq 0 \end{aligned}$$

We call *supply correspondence*, $y(p)$ the production plan that solves PMP, and *profit function*, $\pi(p)$, the value function of problem.

A closer look at the Profit Maximization Problem

- The optimizer of PMP gives to the firm the highest possible profits, given the technological constraint ($f(z)$).
- It corresponds to a point which simultaneously belongs to an iso-profit curve and to the production set.
- Specifically, any production plan that solves PMP, $y^* \in y(p)$, is a point at which the highest iso-profit curve of the firm is tangent to the boundary of the production set, identified by the production function.

Profit Maximization: single output

- Suppose $f(\cdot)$ is a differentiable function, the necessary first order conditions for z^* to solve the PMP are

$$p \frac{\partial f(z^*)}{\partial z_k} \leq w_k \quad \text{for every } k = 1, \dots, m-1 \quad (1)$$

with (1) holding as an equality if $z_k^* > 0$.

Profit Maximization: single output

- The first order conditions in (1) imply that at the solution z^* , for every pair of inputs l, k

$$MRTS_{l,k}(z^*) = \frac{w_l}{w_k}$$

- That is, at the optimum, the marginal rate of technical substitution be equal to the ratio of input prices, which is the economic rate of substitution, for every pair of inputs.
- If Y is convex, i.e. the production function is concave, the first-order conditions are *necessary and sufficient* for identifying the profit-max production plan.

Theorem 1

Suppose that Y is non-empty, closed and satisfies free-disposal. Let $\pi(p)$ be the profit function and $y(p)$ be the supply correspondence of the profit maximization problem for Y , then

- i) $\pi(p)$ is homogeneous of degree one,
i.e. $\pi(\alpha p) = \alpha \pi(p)$ for every $\alpha > 0$;
- ii) $\pi(p)$ is convex;
- iii) if Y is convex, then $Y = \{y \in \mathbb{R}^m : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$;
- iv) $y(p)$ is homogeneous of degree zero;
i.e. $y(\alpha p) = y(p)$ for every $\alpha > 0$;

Theorem 1- continued

- v) if Y is convex, $y(p)$ is a convex set for all p . If Y is strictly convex, $y(p)$ is a singleton;
- vi) [*Hotelling's lemma*] If $y(\bar{p})$ is a singleton, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla \pi(\bar{p}) = y(\bar{p})$;
- vii) If $y(\bar{p})$ is a function and it is differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semi-definite matrix, with $Dy(\bar{p}) \cdot \bar{p} = 0$.

Theorem 1

Suppose that Y is non-empty, closed and satisfies free-disposal. In the profit maximisation problem,

- i) $\pi(p)$ is homogeneous of degree one, i.e. $\pi(\alpha p) = \alpha \pi(p)$ for every $\alpha > 0$;
- ii) $\pi(p)$ is convex;
- iii) if Y is convex, then $Y = \{y \in \mathbb{R}^m : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$;
- iv) $y(p)$ is homogeneous of degree zero; i.e. $y(\alpha p) = y(p)$ for every $\alpha > 0$;
- v) if Y is convex, and $f(\cdot)$ is quasi-concave, then $y(p)$ is a convex set for all p . If Y is strictly convex, $y(p)$ is a singleton;
- vi) If $y(\bar{p})$ is a singleton, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla \pi(\bar{p}) = y(\bar{p})$;
- vii) If $y(\bar{p})$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2 \pi(\bar{p})$ is a symmetric and positive semi-definite matrix, with $Dy(\bar{p}) \cdot \bar{p} = 0$.

The profit function and the supply correspondence

Proof of Theorem 1

- i) Homogeneity of degree one of $\pi(p)$ follows immediately from the linearity of the profit function.
- iii) $\pi(p)$ gives a way to characterize the technology through the profit function.
- iv) If all the input prices and the output one vary by the same proportion, the firm's optimal demands of inputs and supplies of outputs do not change ($y(p)$ is homogeneous of degree zero).

The profit function and the supply correspondence

Proof of Theorem 1- ii) $\pi(p)$ is convex

- $\pi(p)$ convex in p implies that, for every pair (p, p') and $\alpha \in [0, 1]$,

$$\pi(\alpha p + (1 - \alpha)p') \leq \alpha\pi(p) + (1 - \alpha)\pi(p')$$

- Take two price profiles, p and p' and supply correspondences, $y(p)$ and $y(p')$, respectively. Consider their convex combination $\alpha p + (1 - \alpha)p'$, and an optimal production plan at that combination,
 $\tilde{y} \in y(\alpha p + (1 - \alpha)p')$
- Evaluate the value of the profit function at $\tilde{y}$,
$$\pi(\alpha p + (1 - \alpha)p') = (\alpha p + (1 - \alpha)p')\tilde{y} = \alpha p\tilde{y} + (1 - \alpha)p'\tilde{y} \leq \alpha p y(p) + (1 - \alpha)p' y(p') = \alpha\pi(p) + (1 - \alpha)\pi(p'). \blacksquare$$

The profit function and the supply correspondence

Proof of Theorem 1

- v) Uniqueness (multiplicity) result parallels what we found for the consumer problem(s); here it depends on the convexity of Y or concavity of $f(\cdot)$ in single-output models.
- vi) Same idea of the Shephard's Lemma.
- vii) The positive semi-definiteness of $D^2\pi(p)$ identifies the **law of supply**, i.e., the own price effects on the (net) supply correspondence of each good are non-negative. That is, if the price of an output increases, its (optimal) supply increases, and if the price of an input increases, its (optimal) demand decreases.

- Notice that when solving PMP the firm looks for the least costly combination of the inputs that allows to produce the optimal levels of output goods, given her technology
- Hence, as an implication of choosing the profit-maximizing production plan, the neoclassical firm minimizes her costs of production
- An alternative way of analysing profit maximisation is to first have the firm select its cost-minimising factor inputs and then determine its scale of production given the output market conditions.

- Two stage problem. First, the firm determines its cost function that summarises conditions on the input markets. Then, it selects the optimal quantity of output as a function of the input market prices and its technology (embedded in the cost function), and the output market price.
- Consider the single-output case, and let us now formulate the cost-minimization problem of the firm when p is the price of the output, and $w = (w_1, \dots, w_{m-1})$ is the vector of the $(m - 1)$ input prices

Cost Minimization Problem: single-output case

- Fix the level of output at q , and consider the following problem

$$\begin{aligned} \max_z \quad & (w_1 z_1 + \dots + w_{m-1} z_{m-1}) \\ \text{s.t.} \quad & f(z_1, \dots, z_{m-1}) \geq q \end{aligned}$$

- This is equivalent to

$$\begin{aligned} \min_z \quad & (w_1 z_1 + \dots + w_{m-1} z_{m-1}) \\ \text{s.t.} \quad & f(z_1, \dots, z_{m-1}) \geq q \end{aligned}$$

Cost Minimisation Problem: single-output case

- At fixed q , the cost minimisation problem (CMP) is

$$\begin{aligned} \min_z \quad & w_1 z_1 + \dots + w_{m-1} z_{m-1} \\ \text{s.t.} \quad & f(z_1, \dots, z_{m-1}) \geq q \end{aligned}$$

- The solution to CMP is a profile of inputs $(z_1^*, \dots, z_{m-1}^*)$ that minimizes the firm's cost of production, for given q and input prices $w = (w_1, \dots, w_{m-1})$
- Each $z_i^*(w, q)$ is the *conditional factor demand* for input i at prices w and production level q
- The value of the problem is the *cost function*, labelled $C(w, q)$.

Cost Minimization: single output

Characterization by Lagrange method

- Suppose $f(\cdot)$ is differentiable, the necessary first order conditions for a solution to such optimization problem are

$$w_k \geq \lambda \frac{\partial f(z^*)}{\partial z_k} \quad \text{for every } k = 1, \dots, m-1 \quad (2)$$

with equality if $z_k^* > 0$

- If Y is convex, i.e., $f(\cdot)$ is concave, these first-order conditions are *necessary and sufficient* for identifying the cost-minimising vector of inputs.

Cost Minimization: single output

Optimality conditions

- As for the PMP, the first order conditions in (2) imply that at the solution z^* , for every pair of inputs l, k

$$MRTS_{l,k}(z^*, q) = \frac{w_l}{w_k}$$

- That is, at the optimum, the marginal rate of technical substitution be equal to the ratio of input prices, which is the economic rate of substitution, for every pair of inputs
- The marginal rate of technical substitution is the slope of the isoquant associated to the production level q

Cost Minimization: single output

Lagrange multiplier

- Interpret the Lagrange multiplier as the marginal value of relaxing the constraint of CMP, i.e., $f(z) \geq q$
- Specifically, λ now measures the marginal cost of production, $\frac{\partial C(w,q)}{\partial q}$
(See proof in the Note)

Theorem 2

Suppose that Y is non-empty, closed and satisfies free-disposal. Let $C(w, q)$ be the cost function and $z(w, q)$ be the conditional factor demands for the $(m - 1)$ inputs of the cost minimization problem for a single-output technology Y with production function f , then

- i) $C(w, q)$ is homogeneous of degree one in w , and non-decreasing in q ;
- ii) $C(w, q)$ is concave in w ;
- iii) if the set $\{z \geq 0 : f(z) \geq q\}$ is convex for every q , then
 $Y = \{(-z, q) \in \mathbb{R}^m : w \cdot z \geq C(w, q) \text{ for all } w \gg 0\}$;
- iv) $z(w, q)$ is homogeneous of degree zero in w ;

Theorem 2- continued

- v) if the set $\{z \geq 0 : f(z) \geq q\}$ is convex, $z(w, q)$ is a convex set. If the set $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, $z(w, q)$ is a singleton;
- vi) [*Shepard's lemma*] If $z(w, q)$ is a singleton, then $C(\cdot)$ is differentiable at $\bar{w}$ and $\nabla_w C(\bar{w}, q) = z(\bar{w}, q)$;
- vii) If $z(w, q)$ is a function, differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 C(\bar{w}, q)$ is a symmetric and negative semi-definite matrix, with $D_w z(\bar{w}, q) \cdot \bar{w} = 0$;

Theorem 2- continued

- viii) If $f(\cdot)$ is a homogeneous of degree one, i.e., it exhibits constant returns to scale, then $z(\cdot)$ and $C(\cdot)$ are homogeneous of degree one in q ;
- ix) If $f(\cdot)$ is concave, then $C(\cdot)$ is a convex function of q , and the marginal costs are non-decreasing in q .

Cost function and conditional factor demands: properties

Proof of Theorem 2

- i) Total costs linearly increase in input prices.
- iii) It is a way to characterize the technology through the cost function, parallel to PMP, and it builds a dual relationship between costs and profits.
- iv) If all the input prices vary by the same proportion, the firm's optimal demands of inputs do not change.

Cost function and conditional factor demands: properties

Proof of Theorem 2

- v) Parallel to what we found for the consumer problem(s), and for PMP, stated for the relevant production set.
- vi) Same idea of the Shephard's Lemma in consumption.
- vii) The negative semi-definiteness of $D_w^2 \pi(p)$ identifies the **law of supply** for the conditional factor demands, , i.e., if the price of an input increases, its (optimal) demand decreases.

Profit Maximization with cost function: single output

- Once the cost minimization problem has been solved, the firm has to determine the optimal level of output q , by solving the following problem

$$\max_q pq - C(w, q)$$

- The solution of this problem is a level of q^* that maximizes profits while minimizing the firm's cost of production, given the technology $f(\cdot)$ and the input prices $(w_1, \dots, w_{m-1})$

Profit Maximization with cost function: single output

- The necessary first order conditions for a solution to such optimization problem are

$$p - \frac{\partial C(w, q)}{\partial q} = 0 \quad (3)$$

with equality if $q > 0$

- At an interior optimum, price equal marginal cost.
- If $C(w, q)$ is convex in q , i.e., $f(\cdot)$ is concave, this first-order condition is *necessary and sufficient* for identifying q^* , i.e. the firm's optimal output level

Aggregate Supply

- Suppose there are J firms in the economy, each with a given production set $Y_1, \dots, Y_J$, which is non-empty, closed and satisfies free disposal
- The aggregate supply correspondence is the sum of the individual supplies of each firm
- The aggregate profits obtained when each firm maximizes profits independently are the same that would be obtained if the firms were coordinating their actions in a joint profit maximization decision

- Are the optimal production plans of a neoclassical firm efficient?
- A production plan $y \in Y$ is *efficient* if there exists no other production plan $\tilde{y} \in Y$, such that $\tilde{y} \geq y$ and $\tilde{y} \neq y$
- In other words, for a production plan to be efficient there must be no other feasible plan that either generates the same output with less inputs, or that produces more output with the same inputs
- In a single-output model of production, efficient plans are those for which $f(z_1, \dots, z_{m-1}) = q$.

First Theorem of Welfare Economics

First Welfare Theorem

If $y \in Y$ is profit maximizing for some $p \gg 0$, then it is efficient.

Proof. Suppose by contradiction that there is a $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. Since all prices (output and input) are strictly positive this implies that $p \cdot y' > p \cdot y$, which implies that y is not profit maximizing. ■

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First Theorem of Welfare Economics: comments

- The first welfare theorem implies that if many (atomistic) firms independently maximize profits at prices $p \gg 0$, the aggregate result is efficient for the economy
- Having strictly positive prices is essential, otherwise it is possible to find an example of a production plan $y \in Y$ that is profit-maximizing for some $p \geq 0$ (with $p \neq 0$) which is socially inefficient!

Second Welfare Theorem

Assume Y is a convex set. Then, every efficient production plan $y \in Y$ is profit-maximizing for some non-zero price vector $p \geq 0$.

Second Theorem of Welfare Economics

Proof

Proof. Suppose that $y \in Y$ is efficient. Define the set $P_y = \{y' \in \mathbb{R}^m : y' \gg y\}$. P_y is a convex set and because y is efficient (hence, on the frontier of the transformation function), $P_y \cap Y = \emptyset$.

- Using the separating hyperplane theorem, we know that there is some $p \neq 0$ such that $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$
- This implies in particular, that $p \cdot y' \geq p \cdot y$ for every $y' \gg y$
- Therefore, we must have $p \geq 0$ because if $p_k < 0$ for some k , we would have $p \cdot y' < p \cdot y$ for $y' \gg y$ with $y'_k - y_k$ sufficiently large
- Now take any $y'' \in Y$. Then, $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$.
- Because we can choose y' arbitrarily close to y , we can conclude that $p \cdot y \geq p \cdot y''$ for every $y'' \in Y$, that is, y is profit maximizing. ■