Advanced Microconomics 2025/2026

Master of Science in Economics

Problem Set 6 - Mixed exercises

December 14, 2025

Question 1 Let $\mathcal{E} = \{u_i, \omega_i\}_{i=1}^I$ be an exchange economy with L commodities. Show that if the allocation $\overline{x}$ is a solution for

$$\begin{array}{lll} \max_x & \lambda_1 u_1(x_1) + \ldots + \lambda_I u_I(x_I) & \\ s.t. & \sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i & \qquad (PC), \\ & x_i \geq 0, i = 1, \ldots, I & \end{array}$$

where $\lambda_i > 0$ for all $i \in \{1, \ldots, I\}$, then $\overline{x}$ is Pareto efficient.

Solution 3. Suppose, by contradiction, that $\overline{x}$ solves (PC) but it is not Pareto efficient. Then, there exists a feasible allocation $y = (y^1, \ldots, y^I)$ with $y^i \in \mathbb{R}^n_+$ such that $u^i(y^i) \geq u^i(\overline{x}^i)$ for all $i \in \{1, \ldots, I\}$, and $u^i(y^i) > u^i(\overline{x}^i)$ for at least one i. Thus, y satisfies the constraints in (*PC*) and is such that

$$\sum_{i=1}^I \lambda_i u^i(y^i) > \sum_{i=1}^I \lambda_i u^i(\overline{x}^i),$$

i.e. $\overline{x}$ is not a solution to (*PC*), a contradiction. Therefore, we conclude that $\overline{x}$ is Pareto efficient.

Question 2 Consider the $I = L = 2$ exchange economy, with $i = A, B$ where

$$\begin{array}{ll} u_A(x_{11}, x_{21}) = x_{11}x_{21} & \text{and } \omega_1 = (18, 4) \\ u_A(x_{12}, x_{22}) = \ln x_{12} + 2\ln x_{22} & \text{and } \omega_2 = (3, 6). \end{array}$$

1. Find a Walrasian equilibrium of this economy.

2. Find the set of all Pareto efficient allocations.

Solution 2. (a) Solving for the (Walrasian) demands of consumers 1 and 2 yield, respectively

$$x_A(p) = \left(9 + 2\frac{p_2}{p_1}, 9\frac{p_1}{p_2} + 2\right),$$
$$x_B(p) = \left(1 + 2\frac{p_2}{p_1}, 2\frac{p_1}{p_2} + 4\right).$$

Thus, the excess demand vector becomes

$$Z(p) = \left(4\frac{p_2}{p_1} - 11, 11\frac{p_1}{p_2} - 4\right).$$

Solving the system for $Z(p^*) = 0$ implies that

$$\left\{p \gg 0 : p = \lambda\left(1, \frac{11}{4}\right), \lambda > 0\right\}$$

is the set of prices sustaining Walrasian equilibria.
(b) Since the utility functions are strongly increasing, the Pareto efficient allocations can be characterized as the solution to

$$\begin{array}{ll} \max & \ln x_{1B} + 2\ln x_{2B} \\ s.t. & x_{1A}x_{2A} = \overline{u} \\ & x_{1A} + x_{1B} = 21 \\ & x_{2A} + x_{2B} = 10 \\ & x \gg 0 \end{array}$$

for all $u \in (0, U)$, where U denotes consumer 1's utility level when he consumes the aggregate endowment by himself.

The first order conditions to this problem are such that

$$\begin{aligned} x_{1A}x_{2B} &= 2x_{2A}x_{1B}, \\ x_{1A} + x_{1B} &= 21, \\ x_{2A} + x_{2B} &= 10, \\ x_{1A}x_{2A} &= \overline{u}. \end{aligned}$$

Solving the system yields the set of Pareto efficient allocations, namely

$$\begin{aligned} x_{1A} &= \frac{-\overline{u} + \sqrt{\overline{u}^2 + 1680\overline{u}}}{20}, \\ x_{2A} &= \frac{20\overline{u}}{-\overline{u} + \sqrt{\overline{u}^2 + 1680\overline{u}}}, \\ x_{1B} &= 21 - \frac{-\overline{u} + \sqrt{\overline{u}^2 + 1680\overline{u}}}{20}, \\ x_{2B} &= 10 - \frac{20\overline{u}}{-\overline{u} + \sqrt{\overline{u}^2 + 1680\overline{u}}}. \end{aligned}$$

Alternative method.

(b) Pareto set

Because both utility functions are strictly increasing, an allocation is Pareto efficient iff it is feasible and there is no other feasible allocation that makes both agents strictly better off. For interior allocations, this is equivalent to equality of marginal rates of substitution (MRS) plus feasibility.

Step 1: MRS condition. For consumer A,

$$u_A(x_{1A}, x_{2A}) = x_{1A}x_{2A} \quad \Rightarrow \quad MU_{1A} = x_{2A}, \quad MU_{2A} = x_{1A},$$

so

$$MRS_A^{1,2} = \frac{MU_{1A}}{MU_{2A}} = \frac{x_{2A}}{x_{1A}}.$$

For consumer B,

$$u_B(x_{1B}, x_{2B}) = \ln x_{1B} + 2\ln x_{2B} \Rightarrow MU_{1B} = \frac{1}{x_{1B}}, \quad MU_{2B} = \frac{2}{x_{2B}},$$

so

$$MRS_B^{1,2} = \frac{MU_{1B}}{MU_{2B}} = \frac{x_{2B}}{2x_{1B}}.$$

At an interior Pareto efficient allocation we must have

$$MRS_A^{1,2} = MRS_B^{1,2} \quad \Rightarrow \quad \frac{x_{2A}}{x_{1A}} = \frac{x_{2B}}{2x_{1B}} \quad \Leftrightarrow \quad 2x_{2A}x_{1B} = x_{1A}x_{2B}. \tag{$*$}$$

Step 2: Feasibility. Feasibility requires

$$\begin{aligned} x_{1A} + x_{1B} &= 21, \\ x_{2A} + x_{2B} &= 10, \\ x_{1A}, x_{2A}, x_{1B}, x_{2B} &> 0. \end{aligned}$$

Step 3: Contract curve in A's coordinates. Using feasibility, express B's consumption in terms of A's:

$$x_{1B} = 21 - x_{1A}, \qquad x_{2B} = 10 - x_{2A}.$$

Plugging these into the tangency condition $(*)$:

$$\begin{aligned} 2x_{2A}(21 - x_{1A}) &= x_{1A}(10 - x_{2A}) \\ 42x_{2A} - 2x_{1A}x_{2A} &= 10x_{1A} - x_{1A}x_{2A} \\ 42x_{2A} - x_{1A}x_{2A} &= 10x_{1A}. \end{aligned}$$

Rearranging,

$$x_{2A}(42 - x_{1A}) = 10x_{1A} \quad \Rightarrow \quad x_{2A} = \frac{10x_{1A}}{42 - x_{1A}},$$

with $x_{1A} \in (0, 42)$ and feasibility further restricting $x_{1A} \in (0, 21)$, $x_{2A} \in (0, 10)$.

Thus, in (x_{1A}, x_{2A}) space the (interior) Pareto set is

$$\left\{(x_{1A}, x_{2A}) : x_{2A} = \frac{10x_{1A}}{42 - x_{1A}},\ 0 < x_{1A} < 21,\ 0 < x_{2A} < 10\right\}.$$

Question 3 Consider the production economy $\big(\{X_i, \succsim_i\}_{i=1}^I, \{Y_j\}_{j=1}^J, \{\theta_{i1}, \dots, \theta_{iJ}\}_{i=1}^I, (\overline{\omega}_1, \dots, \overline{\omega}_L)\big)$. We say the list $(\overline{x}, \overline{y}, p^*)$, where $(\overline{x}, \overline{y})$ is an allocation and $p^* > 0$ is a price vector, is an equilibrium with transfers if there exists $(w_1, \dots, w_I) \gg 0$ with

$$\sum_{i=1}^I w_i = p^* \sum_{i=1}^I \omega_i + p^* \sum_{j=1}^J \overline{y}_j$$

such that:
(i) for all $j \in \{1, \dots, J\}$, $p^* \cdot y_j \leq p^* \cdot \overline{y}_j$ for all $y_j \in Y_j$;
(ii) for all $i \in \{1, \dots, I\}$, $\overline{x}_i$ is a solution for

$$\begin{array}{ll} \max & u_i(x_i) \\ s.t. & p^* \cdot x_i \leq w_i \\ & x_i \geq 0 \end{array}$$

(iii) $\sum_{i=1}^I \overline{x}_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J \overline{y}_j$.

(a) Show that every Walrasian equilibrium of $\mathcal{E}$ is an equilibrium with transfers.

(b) Suppose the utility functions u^i are locally non–satiated. If $(\overline{x}, \overline{y}, p^*)$ is an equilibrium with transfers, then $(\overline{x}, \overline{y})$ is Pareto efficient.

Solution 3. **(a)** Let $(p^*, \overline{x}, \overline{y})$ constitute a Walrasian equilibrium in the production economy $\mathcal{E}$. Thus, we have that
(i) $p^* \cdot \overline{y}^j \geq p^* \cdot y^j$ for all $y^j \in Y^j$ and $j \in \{1, \dots, J\}$;
(ii) for all $i \in \{1, \dots, I\}$, $\overline{x}^i$ is a solution to

$$\begin{array}{ll} \max_x & u^i(x) \\ s.t. & p^* \cdot x \leq p \cdot e^i + \sum_{j=1}^J \theta^{ij} p^* \cdot \overline{y}^j \\ & x \geq 0; \end{array}$$

(iii) $\sum_{i=1}^I \overline{x}^i = \sum_{i=1}^I e^i + \sum_{j=1}^J \overline{y}^j$.

Therefore, taking $w^i = p \cdot e^i + \sum_{j=1}^J \theta^{ij} p^* \cdot \overline{y}^j$, we have that

$$\begin{aligned} \sum_{i=1}^I w^i &= p^* \cdot \sum_{i=1}^i e^i + \sum_{i=1}^I \sum_{j=1}^J \theta^{ij} p^* \cdot \overline{y}^j \\ &= p^* \cdot \sum_{i=1}^I e^i + p^* \cdot \sum_{j=1}^J \overline{y}^j, \end{aligned}$$

which allows us to conclude that $(\overline{x}, \overline{y}, p^*)$ is a price equilibrium with transfers.
(b) If the allocation $(\overline{x}, \overline{y})$ associated with the price equilibrium with transfers $(p^*, \overline{x}, \overline{y})$ is not Pareto efficient, then there exists an allocation $(\widehat{x}, \widehat{y})$ such that $\sum_{i=1}^I \widehat{x}^i = \sum_{i=1}^I e^i + \sum_{j=1}^J \widehat{y}^j$ and $u^i(\widehat{x}^i) \geq u^i(\overline{x}^i)$ for all $i \in \{1, \dots, I\}$, with strict inequality for at least one consumer.

Because u^i are locally non–satiated, $p^* \cdot \widehat{x}^i \geq p^* \cdot \overline{x}^i = w^i$ for all i, with strict inequality for at least one i. Thus, we must have

$$\sum_{i=1}^{I} p^* \cdot \widehat{x}^i > \sum_{i=1}^{I} w^i = p^* \cdot \sum_{i=1}^{I} e^i + p^* \cdot \sum_{j=1}^{J} \overline{y}^j,$$

which implies that

$$\begin{aligned} & p^* \cdot \sum_{i=1}^{I} e^i + p^* \cdot \sum_{j=1}^{J} \widehat{y}^j > p^* \cdot \sum_{i=1}^{I} e^i + p^* \cdot \sum_{j=1}^{J} \overline{y}^j \\ & \Rightarrow p^* \cdot \sum_{j=1}^{J} \widehat{y}^j > p^* \cdot \sum_{j=1}^{J} \overline{y}^j \end{aligned}$$

and thus, for at least one j, we have $p^* \cdot \sum_{j=1}^{J} \widehat{y}^j > p^* \cdot \sum_{j=1}^{J} \overline{y}^j$. But this contradicts the second condition of a price equilibrium with transfers. Therefore, $(p^*, \overline{x}, \overline{y})$ must be a Pareto efficient allocation.

Question 4 Consider again a 2×2 pure exchange economy with total endowment $(3, 3)$ and endowments

$$\omega_A = (2, 1), \qquad \omega_B = (1, 2).$$

Preferences:

$$u_A(x_{1A}, x_{2A}) = x_{1A}^{2/3} x_{2A}^{1/3}, \qquad u_B(x_{1B}, x_{2B}) = x_{1B}^{1/3} x_{2B}^{2/3}.$$

(a) Compute the MRS of each consumer and derive the equation of the (interior) contract curve in terms of (x_{1A}, x_{2A}).

(b) Derive the individual Cobb–Douglas demands of A and B given prices $p = (p_1, p_2)$ and incomes m_A, m_B.

(c) Find a Walrasian equilibrium price vector (up to normalization) and the corresponding allocation.

(d) Check whether the equilibrium allocation lies on the contract curve.

Solution 4. (a) Contract curve.
For $u_A(x_{1A}, x_{2A}) = x_{1A}^{2/3} x_{2A}^{1/3}$:

$$MU_{1A} = \frac{2}{3} x_{1A}^{-1/3} x_{2A}^{1/3}, \quad MU_{2A} = \frac{1}{3} x_{1A}^{2/3} x_{2A}^{-2/3}.$$

Thus

$$MRS_A^{1,2} = \frac{MU_{1A}}{MU_{2A}} = \frac{\frac{2}{3} x_{1A}^{-1/3} x_{2A}^{1/3}}{\frac{1}{3} x_{1A}^{2/3} x_{2A}^{-2/3}} = 2\,\frac{x_{2A}}{x_{1A}}.$$

For $u_B(x_{1B}, x_{2B}) = x_{1B}^{1/3} x_{2B}^{2/3}$:

$$MU_{1B} = \frac{1}{3} x_{1B}^{-2/3} x_{2B}^{2/3}, \quad MU_{2B} = \frac{2}{3} x_{1B}^{1/3} x_{2B}^{-1/3},$$

so

$$MRS_B^{1,2} = \frac{MU_{1B}}{MU_{2B}} = \frac{\frac{1}{3} x_{1B}^{-2/3} x_{2B}^{2/3}}{\frac{2}{3} x_{1B}^{1/3} x_{2B}^{-1/3}} = \frac{1}{2}\,\frac{x_{2B}}{x_{1B}}.$$

Interior Pareto efficiency requires $MRS_A^{1,2} = MRS_B^{1,2}$:

$$2\frac{x_{2A}}{x_{1A}} = \frac{1}{2}\frac{x_{2B}}{x_{1B}}.$$

Using feasibility, $x_{1B} = 3 - x_{1A}$ and $x_{2B} = 3 - x_{2A}$, so

$$2\frac{x_{2A}}{x_{1A}} = \frac{1}{2}\frac{3 - x_{2A}}{3 - x_{1A}}.$$

Rearranging:

$$4x_{2A}(3 - x_{1A}) = x_{1A}(3 - x_{2A}).$$

Solving for x_{2A} as a function of x_{1A} gives

$$x_{2A} = \frac{x_{1A}}{4 - x_{1A}},$$

for $x_{1A} \in (0,3)$ such that also $x_{2A} \in (0,3)$. This is the (interior) contract curve.

(b) Individual demands.

For Cobb–Douglas $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, the demands are

$$x_1 = \alpha \frac{m}{p_1}, \qquad x_2 = (1-\alpha)\frac{m}{p_2}.$$

Consumer A: $\alpha_A = 2/3$, so

$$x_{1A}(p, m_A) = \frac{2}{3}\frac{m_A}{p_1}, \quad x_{2A}(p, m_A) = \frac{1}{3}\frac{m_A}{p_2}.$$

Consumer B: $\alpha_B = 1/3$, so

$$x_{1B}(p, m_B) = \frac{1}{3}\frac{m_B}{p_1}, \quad x_{2B}(p, m_B) = \frac{2}{3}\frac{m_B}{p_2}.$$

Incomes:

$$m_A = 2p_1 + p_2, \qquad m_B = p_1 + 2p_2.$$

(c) Walrasian equilibrium.

Total endowments: $(3,3)$. Market clearing for good 1:

$$x_{1A}(p) + x_{1B}(p) = 3.$$

Plugging in:

$$\frac{2}{3}\frac{m_A}{p_1} + \frac{1}{3}\frac{m_B}{p_1} = 3 \quad \Rightarrow \quad \frac{1}{3p_1}(2m_A + m_B) = 3.$$

Compute $2m_A + m_B$:

$$2m_A + m_B = 2(2p_1 + p_2) + (p_1 + 2p_2) = 5p_1 + 4p_2.$$

So

$$\frac{1}{3p_1}(5p_1 + 4p_2) = 3 \quad \Rightarrow \quad 5 + 4\frac{p_2}{p_1} = 9 \quad \Rightarrow \quad \frac{p_2}{p_1} = 1.$$

Thus $p_1 = p_2$, and we may normalize $p = (1,1)$.

At $p = (1,1)$, incomes:

$$m_A = 2 \cdot 1 + 1 = 3, \qquad m_B = 1 + 2 \cdot 1 = 3.$$

Demands:

$$x_{1A} = \frac{2}{3} \cdot \frac{3}{1} = 2, \quad x_{2A} = \frac{1}{3} \cdot \frac{3}{1} = 1,$$
$$x_{1B} = \frac{1}{3} \cdot \frac{3}{1} = 1, \quad x_{2B} = \frac{2}{3} \cdot \frac{3}{1} = 2.$$

So the Walrasian equilibrium is

$$p^* = (1,1), \quad x_A^* = (2,1), \quad x_B^* = (1,2),$$

which happens to coincide with the initial endowment.

(d) Is the equilibrium allocation on the contract curve?

Plug $x_A^* = (2,1)$ into the contract-curve formula:

$$x_{2A} = \frac{x_{1A}}{4 - x_{1A}} = \frac{2}{4-2} = 1.$$

So $(2,1)$ lies on the curve; thus the equilibrium allocation is Pareto efficient and on the contract curve.

Question 5 Consider the symmetric Cobb–Douglas economy from the previous exercise:

$$u_A(x_{1A}, x_{2A}) = x_{1A}^{1/2} x_{2A}^{1/2}, \quad u_B(x_{1B}, x_{2B}) = x_{1B}^{1/2} x_{2B}^{1/2},$$

$$\omega_A = (2, 1), \quad \omega_B = (1, 2), \quad \bar{\omega} = (3, 3).$$

(a) Compute each consumer's utility at the initial endowment.

(b) Characterize the contract curve .

(c) Using the definition of the core, argue that in this economy the core coincides with contract curve, i.e. the set of feasible, Pareto efficient and individually rational allocations. Write this set explicitly for the present economy.

Solution 5. (a) Utility at the endowment.

For consumer A, $\omega_A = (2, 1)$:

$$u_A(\omega_A) = (2 \cdot 1)^{1/2} = \sqrt{2}.$$

For consumer B, $\omega_B = (1, 2)$:

$$u_B(\omega_B) = (1 \cdot 2)^{1/2} = \sqrt{2}.$$

(b) Contract curve.

For both consumers:

$$MRS_i^{1,2} = \frac{x_{2i}}{x_{1i}}.$$

Interior Pareto efficiency requires

$$\frac{x_{2A}}{x_{1A}} = \frac{x_{2B}}{x_{1B}}.$$

Using feasibility $x_{1A} + x_{1B} = 3$, $x_{2A} + x_{2B} = 3$, one can show the only way these ratios coincide for all interior allocations is when

$$x_{2A} = x_{1A}, \qquad x_{2B} = x_{1B}.$$

So the contract curve inside the box is the diagonal line $x_{2A} = x_{1A}$ (with the obvious boundary segments if you include corners).

(c) The core.

In a two-consumer exchange economy, the only non-trivial coalitions are $\{A\}$, $\{B\}$ and $\{A, B\}$. By definition:

- Coalition $\{A, B\}$ can block any allocation that is not Pareto efficient, so core allocations must be Pareto efficient.
- Coalition $\{A\}$ can block any allocation x where $u_A(x_A) < u_A(\omega_A)$, since A can always deviate alone and consume ω_A. Similarly for B.

Hence the core is the set of allocations that are:

1. Feasible ($x_{1A} + x_{1B} = 3$, $x_{2A} + x_{2B} = 3$),

2. Pareto efficient (on the contract curve),

3. Individually rational ($u_A(x_A) \geq u_A(\omega_A)$ and $u_B(x_B) \geq u_B(\omega_B)$).

On the contract curve we can parameterize feasible allocations by $x_A = (t, t)$ and $x_B = (3-t, 3-t)$ with $t \in (0, 3)$. Utilities:

$$u_A(t, t) = t, \qquad u_B(3-t, 3-t) = 3-t.$$

Individual rationality requires:

$$t \geq \sqrt{2}, \quad 3-t \geq \sqrt{2} \quad \Rightarrow \quad \sqrt{2} \leq t \leq 3-\sqrt{2}.$$

Thus the core is the set

$$\left\{ \big(x_A, x_B\big) : x_A = (t, t),\ x_B = (3-t, 3-t),\ \sqrt{2} \leq t \leq 3-\sqrt{2} \right\}.$$

The Walrasian equilibrium allocation $t^* = 3/2$ lies in this interval, and therefore in the core.

Question 6 Consider a pure exchange economy with two consumers A and B and two goods. Total endowment is $\bar{\omega} = (2, 2)$ which is distributed as follows,

$$\omega_A = (2, 2), \qquad \omega_B = (0, 0).$$

Consumers' preferences are represented by the utility functions:

$$u_A(x_{1A}, x_{2A}) = \min\{x_{1A}, 1\} + \min\{x_{2A}, 1\}, \qquad u_B(x_{1B}, x_{2B}) = x_{1B} + x_{2B}.$$

(a) Show that for any strictly positive price vector $p \gg 0$, the allocation where each consumer consumes her endowment $x_A = \omega_A = (2, 2)$ and $x_B = \omega_B = (0, 0)$ is a competitive (Walrasian) equilibrium.

(b) Show that this competitive equilibrium allocation is *not* Pareto efficient. Find the set of all Pareto efficient allocations of this economy.

(c) Why does the First Welfare Theorem fail in this economy? Briefly explain how this connects to your findings in parts (a) and (b).

Solution 6. (a) Competitive equilibrium at the endowment.
Let $p = (p_1, p_2) \gg 0$.

Consumer B. Income:

$$m_B = p \cdot \omega_B = p_1 \cdot 0 + p_2 \cdot 0 = 0.$$

Her budget set is

$$B_B(p) = \{x_B \in \mathbb{R}^2_+ : p_1 x_{1B} + p_2 x_{2B} \leq 0\} = \{(0, 0)\}.$$

So her *only* affordable bundle is $(0, 0)$, which is trivially utility-maximizing:

$$x_B^* = (0, 0) = \omega_B.$$

Consumer A. Income:

$$m_A = p \cdot \omega_A = 2p_1 + 2p_2.$$

Her budget set is

$$B_A(p) = \left\{x_A \in \mathbb{R}^2_+ : p_1 x_{1A} + p_2 x_{2A} \leq 2p_1 + 2p_2\right\}.$$

Her utility is

$$u_A(x_{1A}, x_{2A}) = \min\{x_{1A}, 1\} + \min\{x_{2A}, 1\}.$$

Notice that

$$\min\{x_{1A}, 1\} \leq 1, \quad \min\{x_{2A}, 1\} \leq 1 \quad \Rightarrow \quad u_A(x_{1A}, x_{2A}) \leq 2.$$

Moreover, $u_A(x_{1A}, x_{2A}) = 2$ if and only if $x_{1A} \geq 1$ and $x_{2A} \geq 1$.

Thus the *maximum* utility that A can ever reach is 2.

The endowment bundle is $x_A = \omega_A = (2, 2)$. It is affordable because

$$p_1 \cdot 2 + p_2 \cdot 2 = 2p_1 + 2p_2 = m_A.$$

At $(2,2)$ we have $x_{1A} \geq 1$ and $x_{2A} \geq 1$, hence

$$u_A(2,2) = \min\{2,1\} + \min\{2,1\} = 1 + 1 = 2,$$

which is the maximum feasible utility.

Therefore $(2,2)$ is a utility-maximizing bundle for A at prices p.

Market clearing. With $x_A = (2,2)$ and $x_B = (0,0)$,

$$x_A + x_B = (2,2) = \bar{\omega},$$

so markets clear.

Thus for any $p \gg 0$, the pair $(p,(x_A,x_B)) = (p,(\omega_A,\omega_B))$ is a Walrasian equilibrium.

(b) Pareto inefficiency and the Pareto set.

Step 1. Show that the equilibrium allocation is not Pareto efficient.

Consider the feasible allocation

$$x'_A = (1,1), \qquad x'_B = (1,1).$$

Feasibility:

$$x'_A + x'_B = (1+1, 1+1) = (2,2) = \bar{\omega}.$$

Utilities:

$$u_A(x'_A) = \min\{1,1\} + \min\{1,1\} = 1 + 1 = 2,$$

while at the equilibrium allocation $x_A = (2,2)$ we have

$$u_A(x_A) = \min\{2,1\} + \min\{2,1\} = 1 + 1 = 2.$$

Thus A is *indifferent*: $u_A(x'_A) = u_A(x_A)$.

For consumer B:

$$u_B(x'_B) = 1 + 1 = 2 > 0 = u_B(x_B).$$

So x' is a *Pareto improvement* over the equilibrium allocation: A is no worse off, B is strictly better off. Hence the equilibrium allocation $(x_A,x_B) = ((2,2),(0,0))$ is not Pareto efficient.

Step 2. Characterize the Pareto set.

Let (x_A,x_B) be any feasible allocation:

$$x_{1A}, x_{2A}, x_{1B}, x_{2B} \geq 0, \quad x_{1A} + x_{1B} = 2, \quad x_{2A} + x_{2B} = 2.$$

We first show that in any Pareto efficient allocation we must have

$$x_{1A} \leq 1 \quad \text{and} \quad x_{2A} \leq 1.$$

Suppose instead that $x_{1A} > 1$ and $x_{1B} > 0$. Then we can transfer a small amount $\varepsilon > 0$ of good 1 from A to B, with

$$0 < \varepsilon \leq \min\{x_{1A} - 1, x_{1B}\}.$$

Define a new allocation y by

$$y_{1A} = x_{1A} - \varepsilon, \quad y_{2A} = x_{2A}, \quad y_{1B} = x_{1B} + \varepsilon, \quad y_{2B} = x_{2B}.$$

Feasibility is preserved, and for consumer A:

$$u_A(y_A) = \min\{y_{1A}, 1\} + \min\{y_{2A}, 1\} = \min\{x_{1A} - \varepsilon, 1\} + \min\{x_{2A}, 1\} = 1 + \min\{x_{2A}, 1\} = u_A(x_A),$$

since $x_{1A} - \varepsilon \geq 1$ by construction.

For consumer B:

$$u_B(y_B) = (x_{1B} + \varepsilon) + x_{2B} = u_B(x_B) + \varepsilon > u_B(x_B).$$

Thus y is a Pareto improvement over x, contradicting Pareto efficiency. So in any Pareto efficient allocation we cannot have both $x_{1A} > 1$ and $x_{1B} > 0$. But $x_{1B} > 0$ must hold in any non-degenerate allocation (since B has strictly increasing preferences). Therefore we must have $x_{1A} \leq 1$. A symmetric argument for good 2 implies $x_{2A} \leq 1$.

Hence, in any Pareto efficient allocation,

$$0 \leq x_{1A} \leq 1, \quad 0 \leq x_{2A} \leq 1, \qquad x_{1B} = 2 - x_{1A}, \quad x_{2B} = 2 - x_{2A}.$$

Step 3. Show that every such allocation is Pareto efficient.

Take any allocation with

$$0 \leq x_{1A} \leq 1, \quad 0 \leq x_{2A} \leq 1, \qquad x_{1B} = 2 - x_{1A}, \quad x_{2B} = 2 - x_{2A}.$$

Consumer B's utility is strictly increasing in each good, so any Pareto improvement must involve giving *more* of some good to B, i.e. increasing x_{1B} or x_{2B}. By feasibility, this would require decreasing x_{1A} or x_{2A}.

But for consumer A, since $x_{1A} \leq 1$ and $x_{2A} \leq 1$, we have

$$u_A(x_A) = x_{1A} + x_{2A}.$$

If we reduce either x_{1A} or x_{2A}, u_A strictly decreases. So there is no feasible reallocation that raises B's utility without lowering A's utility.

Therefore *every* allocation satisfying

$$0 \leq x_{1A} \leq 1, \quad 0 \leq x_{2A} \leq 1, \quad x_{1B} = 2 - x_{1A}, \quad x_{2B} = 2 - x_{2A}$$

is Pareto efficient.

Conclusion: The Pareto set is

$$\mathcal{P} = \left\{ (x_A, x_B) : 0 \leq x_{1A} \leq 1,\ 0 \leq x_{2A} \leq 1,\ x_B = (2 - x_{1A},\ 2 - x_{2A}) \right\}.$$

The competitive equilibrium allocation $(x_A, x_B) = ((2, 2), (0, 0))$ does not belong to $\mathcal{P}$, hence it is not Pareto efficient.

(c) Which assumption of the FWT fails?

The First Welfare Theorem requires that each consumer's preferences be *locally non-satiated*. Consumer B has locally non-satiated (indeed strictly monotone) preferences, but consumer A has not. Indeed, A has a satiation point at $(1, 1)$.