Practice session 2

Game Theory - MSc EEBL

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Exercise 1. *Cournot Duopoly*

Two firms compete in a market by simultaneously setting the quantities of a (homogeneous) good to produce (q_i, q_j). Each firm faces a *constant marginal cost* $c_k \in \mathbb{R}_+$. The two firms face the *inverse demand function* $P(Q) = a - bQ$, where Q is the aggregate quantity produced and $(a, b) \in \mathbb{R}^2_+$ are demand parameters. Payoffs are given by each firm's profits.

1. Describe the game as a normal-form game (Players, Strategies, Payoffs).
2. Write the maximization problem for each firm.
3. Solve the maximization problem for each firm, obtaining its *reaction function*. (*Hint: Take the first derivative of profits with respect to each firm's quantity, taking the quantity produced by the other as given).*
4. Find the *NE* of the game, i.e., equilibrium quantities for both firms (*Hint: Use the two reaction functions and solve for* $q_i^\star$).
5. Comment on the equilibrium quantities when a, c_1 and c_2 vary.
6. Find the payoffs $\pi_i(q_i^\star, q_j^\star)$ for $i = 1, 2$ obtained by firms when they play this *NE* (*Hint: Replace equilibrium quantities in profits*).
7. Now suppose that both firms also face a *fixed cost* $F = 1$. Does this affect the equilibrium quantity? Does this change equilibrium payoffs? Explain.

Answer of Exercise 1.

1. • Players: $N = \{\text{Firm } 1, \text{Firm } 2\}$.

- Strategies: $S_1 = S_2 = [0, \infty)$.
- Payoffs:

$$\pi_1(q_1, q_2) = (a - b(q_1 + q_2) - c_1)q_1$$
$$\pi_2(q_1, q_2) = (a - b(q_1 + q_2) - c_2)q_2$$

2. When you write the maximization problem for Firm 1, remember to take the quantity produced by Firm 2 as given.

$$\max_{q_1} \ \pi_1(q_1, q_2) = (a - b(q_1 + q_2) - c_1)q_1,$$
$$\max_{q_2} \ \pi_2(q_1, q_2) = (a - b(q_1 + q_2) - c_2)q_2.$$

3. For each firm, we compute the *first-order condition* and then solve for quantity.

   - Firm 1:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1} = 0 \implies a - 2bq_1 - bq_2 - c_1 = 0 \implies q_1 = \frac{a - c_1 - bq_2}{2b}.$$

   - Firm 2:

$$\frac{\partial \pi_2(q_1, q_2)}{\partial q_2} = 0 \implies a - 2bq_2 - bq_1 - c_2 = 0 \implies q_2 = \frac{a - c_2 - bq_1}{2b}.$$

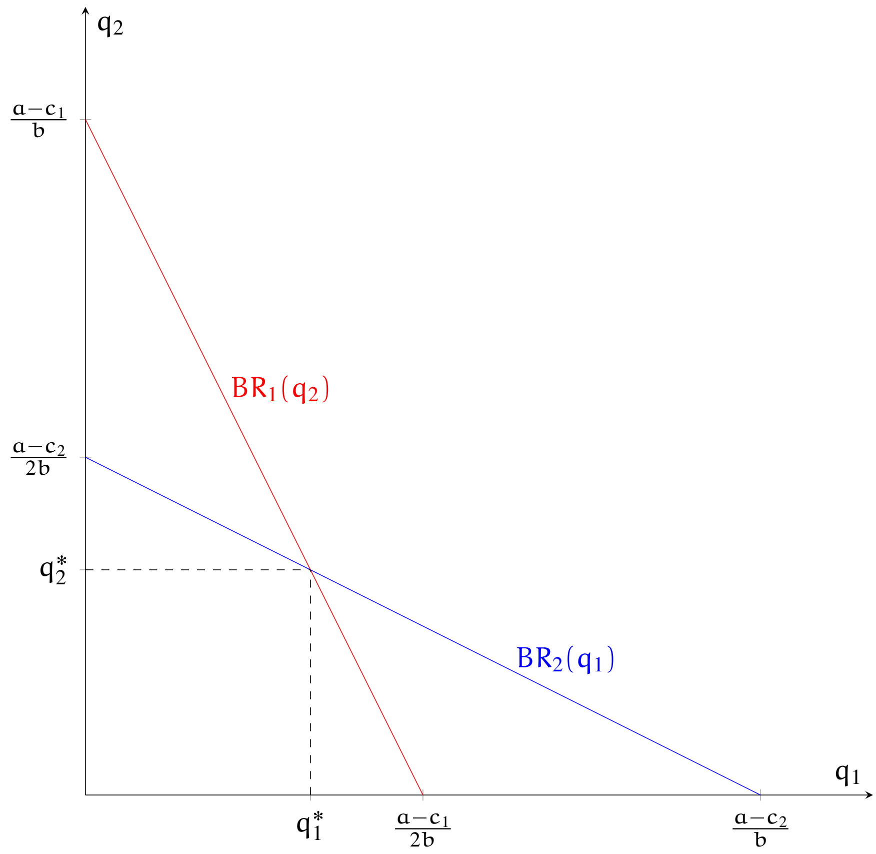
   The reaction functions therefore write:

$$\begin{cases} BR_1(q_2) & = \dfrac{a - c_1 - bq_2}{2b} \\ BR_2(q_1) & = \dfrac{a - c_2 - bq_1}{2b} \end{cases}$$

To illustrate the best-response functions and the Nash equilibrium we can plot the former in the plane (q_1, q_2). Notice that we can directly use the expression of $BR_2(q_1)$ and plot it as it is clearly a function of q_1. To plot $BR_1(q_2)$, however, we have to invert it so that we obtain q_2 as a function of q_1 as well. Formally, we have to solve the following equation for q_2:

$$q_1 = \frac{a - c_1 - bq_2}{2b}.$$

This immediately gives $q_2(q_1) = \frac{a - c_1}{b} - 2q_1$. Plotting the two functions in the plane (q_1, q_2) gives



Graphically, the Nash equilibrium is the point (q_1^*, q_2^*) at which the two best-response functions intersect.

4. At a Nash equilibrium, we know that each firm chooses the best strategy given the strategy of the other firm. Here the optimal strategy of firm 1 is $BR_1(q_2)$ and that of firm 2 is $BR_2(q_1)$ and they obviously depends on the strategy of the other firm. So when firm 1 chooses some q_1, firm 2 will strategically choose q_2 according to $BR_2(q_1)$. But then, firm 1 will strategically choose q_1 according to $BR_1(q_2)$, that is, firm 1 will choose $BR_1(BR_2(q_1))$. And so on.

   A situation is a Nash equilibrium when each firm's best response to the other firm's strategy coincide with its own strategy (mathematically it is a fixed point), that is,

$$\begin{cases} BR_1(BR_2(q_1^*)) & = q_1^* \\ BR_2(BR_1(q_2^*)) & = q_2^* \end{cases}$$

   We then solve the above linear system. For instance, plug $BR_2(q_1)$ into firm 1's reaction function $BR_1(q_2)$ and solve for q_1:

$$BR_1(BR_2(q_1)) = \frac{a-c_1}{2b} - \frac{1}{2}\left(\frac{a-c_2-bq_1}{2b}\right)$$

Recall that we must have $BR_1(BR_2(q_1)) = q_1$ so that we have to solve

$$\frac{a-c_1}{2b} - \frac{1}{2}\left(\frac{a-c_2-bq_1}{2b}\right) = q_1$$
$$\Leftrightarrow q_1^* = \frac{a-2c_1+c_2}{3b}.$$

Plugging this value into $BR_2(q_1^*)$ immediately gives

$$q_2^* = \frac{a-2c_2+c_1}{3b}.$$

5. Equilibrium quantities both depend on a, b, c_1 and c_2.

   They are both increasing in a, which is natural as a is the intercept parameter of the demand: A higher a means that there is a higher demand and so firms can sell more quantity.

   Notice that when $c_1 = c_2 \equiv c$, equilibrium quantities are the same and write $q_1^* = q_2^* = \frac{a-c}{3b}$. However, when $c_1 > c_2$ we obtain that $q_1^* < q_2^*$, that is, the most efficient firm produces more at equilibrium.

6. Profits are given by

$$\pi_1(q_1^*, q_2^*) = \frac{(a-2c_1+c_2)^2}{9b},$$
$$\pi_2(q_1^*, q_2^*) = \frac{(a-2c_2+c_1)^2}{9b}.$$

7. The maximization problem for the two firms becomes:

$$\max_{q_1} \pi_1(q_1, q_2) = (a - b(q_1+q_2) - c_1)q_1 - F,$$
$$\max_{q_2} \pi_2(q_1, q_2) = (a - b(q_1+q_2) - c_2)q_2 - F.$$

   Notice that F disappears when taking the first-order condition. As a consequence, NE quantities (and price) are not affected. However, NE profits decrease by 1, the exact amount of the fixed cost.

Exercise 2. *Rock paper scissors*

Pat and Carl meet to play the famous game *Rock paper scissors*. According to this game, both players simultaneously choose between rock, paper, or scissors. Not surprisingly, rock beats scissors, scissors beat paper, and paper beats rock. If a player wins, they get 1 Euro from the other player. If they loose, they pay 1 Euro to the other player. If both players choose the same action, then they both get nothing.

1. Fill the *payoff matrix* using the available information and write the game as a *normal-form* game.

2. Is there any strictly *dominated strategy* for players?

3. Find the *pure-strategy Nash equilibria (NE)* of this game, if any.

4. Find the *mixed-strategy* NE of this game, if any.

Answer of Exercise 2.

1. The payoff matrix is as follows.

			Carl	
		Rock	Paper	Scissors
	Rock	(0,0)	(-1,1)	(1,-1)
Pat	Paper	(1,-1)	(0,0)	(-1,1)
	Scissors	(-1,1)	(1,-1)	(0,0)

To write the game as a normal-form game we have to define:

$$N = \{\text{Pat, Carl}\},$$
$$S_P = S_C = \{\text{Rock, Paper, Scissors}\}.$$

Payoffs from each combination of the two Firms' strategies are given in the matrix.

2. There are **no strictly dominated strategies** for Pat and Carl (*zero-sum game*).

			Carl	
		y_1 Rock	y_2 Paper	$1-y_1-y_2$ Scissors
	x_1 Rock	(0,0)	$(-1,\underline{1})$	$(\underline{1},-1)$
Pat	x_2 Paper	$(\underline{1},-1)$	(0,0)	$(-1,\underline{1})$
	$1-x_1-x_2$ Scissors	$(-1,\underline{1})$	$(\underline{1},-1)$	(0,0)

3. There is **no pure-strategy NE in this game** (*zero-sum game*).

4. To find the *mixed-strategy NE* of this game, compute each player's expected payoff from playing pure strategies when the other is playing respectively the mixed strategies (x_1, x_2) and (y_1, y_2). For Pat:

$$\begin{cases} v_P(\text{Rock}, (y_1, y_2)) = 0 \cdot y_1 - y_2 + (1 - y_1 - y_2) = \mathbf{1 - y_1 - 2y_2}. \\ v_P(\text{Paper}, (y_1, y_2)) = y_1 + 0 \cdot y_2 - 1 + y_1 + y_2 = \mathbf{2y_1 + y_2 - 1}. \\ v_P(\text{Scissors}, (y_1, y_2)) = -y_1 + y_2 + 0 \cdot (1 - y_1 - y_2) = \mathbf{-y_1 + y_2}. \end{cases}$$

The next step is to compute y_1^*, y_2^* such that Pat is indifferent between playing any of her pure strategies and thus willing to randomize over them. To this purpose, equate the expected payoffs from playing any two pure strategies (for instance Rock and Scissors):

$$1 - y_1 - 2y_2 = -y_1 + y_2 \implies 3y_2 = 1 \implies y_2^* = \frac{1}{3}.$$

Use the equality between the expected payoffs from paper and scissors to find y_1^*:

$$2y_1 + y_2 - 1 = -y_1 + y_2 \implies 3y_1 = 1 \implies y_1^* = \frac{1}{3}.$$

Pat will be willing to randomize over her pure strategies if Carl plays Rock, Paper, and Scissors with probability of one third each. For Carl:

$$\begin{cases} v_C((x_1, x_2), \text{Rock}) = 0 \cdot x_1 - x_2 + (1 - x_1 - x_2) = \mathbf{1 - x_1 - 2x_2}. \\ v_C((x_1, x_2), \text{Paper}) = x_1 + 0 \cdot x_2 - 1 + x_1 + y_2 = \mathbf{2x_1 + x_2 - 1}. \\ v_C((x_1, x_2), \text{Scissors}) = -x_1 + x_2 + 0 \cdot (1 - x_1 - x_2) = \mathbf{-x_1 + x_2}. \end{cases}$$

The next step is to compute x_1^*, x_2^* such that Carl is indifferent between playing any of his pure strategies and thus willing to randomize over them. To this purpose, equate the expected payoffs from playing any two pure strategies (for instance Rock and Scissors):

$$1 - x_1 - 2x_2 = -x_1 + x_2 \implies 3x_2 = 1 \implies x_2^* = \frac{1}{3}.$$

Use the equality between the expected payoffs from paper and scissors to find x_1^*:

$$2x_1 + x_2 - 1 = -x_1 + x_2 \implies 3x_1 = 1 \implies x_1^* = \frac{1}{3}.$$

Carl will be willing to randomize over her pure strategies if Carl plays Rock, Paper, and Scissors with probability of one third each. The only *NE* of this game is in *mixed strategies* and it can be expressed as follows:

$$\left(\left(\frac{1}{3},\frac{1}{3}\right);\left(\frac{1}{3},\frac{1}{3}\right)\right) \text{ or } \left(\frac{1}{3}\text{Rock}+\frac{1}{3}\text{Paper}+\frac{1}{3}\text{Scissors},\frac{1}{3}\text{Rock}+\frac{1}{3}\text{Paper}+\frac{1}{3}\text{Scissors}\right).$$

Exercise 3. *Setting a Standard*

A new type of consumer product is about to be introduced in a market in which two firms are active (for example, a video game). The two firms own *competing technologies* (for example, two game consoles) that can be used to run this product, and would like their technology to be the *standard* in the market.

Each firm would prefer its technology to be used exclusively to run the product, as this would increase its sales. In particular, each firm has a payoff of zero if no standard is set (both firms use their own technology). If only one firm's technology is adopted as a standard, that firm gets a payoff of 2, and the other gets 1. Finally, if both firms employ the other firm's technology, they both get a payoff of 0.

1. Fill the payoff matrix using the available information and write the game as a normal-form game.
2. Is there any strictly *dominated strategy* for the two players?
3. Find the *pure-strategy Nash equilibria (NE)* of this game, if any.
4. Find the *mixed-strategy NE* of this game, if any.
5. Now, suppose that firm 1 has a *superior technology*, that is, the latter gets a payoff of 3 when it manages to set the standard. Does this affect the *pure and mixed-strategy NE*? Explain (*Hint: payoffs do not change for firm 2*).

Answer of Exercise 3.

1. The payoff matrix is as follows.

		Firm 2	
		Own Tech.	Comp Tech.
Firm 1	Own Tech.	(0,0)	(2,1)
	Comp Tech.	(1,2)	(0,0)

As usual, the normal form of the game consists of the set of players, the set of of strategies and the payoffs.

$$N = \{\text{Firm1, Firm 2}\},$$
$$S_1 = S_2 = \{\text{Own Tech., Comp Tech.}\}.$$

Payoffs from each combination of the two Firms' strategies are given by the matrix.

2. There is **no strictly dominated pure strategy** for the two firms. In fact *Own Tech.* yields a higher payoff when the other Firm plays *Comp Tech.* ($2 > 0$), but *Comp Tech.* yields a higher payoff when the other Firm plays *Own Tech.* ($1 > 0$).

3. As usual, we underline the best response of each player and this eventually yields the pure-strategy Nash Equilibria (when there is at least one of course).

		Firm 2	
		y Own Tech.	$1-y$ Comp Tech.
Firm 1	Own Tech. x	(0,0)	$(\underline{2},\underline{1})$
	Comp Tech. $1-x$	$(\underline{1},\underline{2})$	(0,0)

There are **two pure-strategy NE** in this game, i.e. $\{\text{Comp Tech., Own Tech.}\}$ and $\{\text{Own Tech., Comp Tech.,}\}$. A standard will be adopted in the industry.

4. Let us denote by $u_i(s_1, s_2)$ the utility of firm $i = 1, 2$ when they play $s_1 \in S_1$ and $s_2 \in S_2$, respectively.

   By a slight abuse of notation, let us say that if we write $u_i(s_1, y)$ with $y \in [0, 1]$, it means that Firm 2 is randomizing over strategies and assigns a probability of y (resp. $1-y$) to strategy "Own Tech." (resp. "Comp Tech."). Similarly, $u_i(x, s_2)$ with $x \in [0, 1]$ means that firm 1 is randomizing over strategies, that is, it plays "Own Tech." with probability x and "Comp Tech."with probability $1-x$.

   Notice that $u_i(s_1, 1) = u_i(s_1, \text{Own Tech.})$ and $u_i(s_1, 0) = u_i(s_1, \text{Comp Tech.})$, that is, when y is either 1 or 0 (degenerate probability distribution), we come back to the case in which Firm 2 play a pure-strategy. A similar observation applies to Firm 1's mixed strategy.

   Let us know look for mixed-strategy equilibria. As when we look for pure-strategy Nash equilibria, we are going to fix one player's strategy, say Firm 2, and look what would the other do in that case, sya Firm 1. The difference, however, is that we are going fix a mixed strategy for Firm 2 rather than a pure strategy.

   In practice, we therefore assume that Firm 2 is playing the mixed strategy $y \in [0, 1]$, that is, Firm 2 plays "Own Tech." with probability y and "Comp Tech." with probability $1-y$.

Then what would Firm 1 do? Firm 1 could choose to play “Own Tech.” ($x = 1$) or “Comp Tech.” ($x = 0$) for sure or could also choose to randomize ($x \in (0,1)$). Let us focus on this last case, that is, when Firm 1 also chooses to randomize (we will see the argument for $x = 1$ and $x = 0$ after).

If we assume that Firm 1 randomizes, then it should be the case that it gets exactly the same payoff when playing “Own Tech.” and when playing “Comp Tech.”. Why? Assume for instance that “Own Tech.” gives a strictly higher payoff to Firm 1, then it would simply always play it, that is it would not randomize (i.e. $x = 1$). Then if we want Firm 1 to actually randomize, none of its strategies should be better than the other.

Then, assuming that Firm 2 plays the mixed strategy $y \in [0,1]$, we can compute Firm 1’s **expected payoffs** from playing Own Tech. and Comp Tech., respectively:

$$\begin{aligned} u_1(\text{Own Tech.}, y) &= 0 \cdot y + 2(1-y) = 2 - 2y, \\ u_1(\text{Comp Tech.}, y) &= y + 0 \cdot (1-y) = y. \end{aligned}$$

For Firm 1 to be willing to randomize over its pure strategies we need:

$$\begin{aligned} u_1(\text{Own Tech.}, y) &= u_2(\text{Comp Tech.}, y) \\ \Leftrightarrow 2 - 2y = y &\Leftrightarrow 3y = 2 \Leftrightarrow y = \frac{2}{3}. \end{aligned}$$

This means that Firm 1 will be indifferent between playing Own Tech. and Comp Tech. only if Firm 2’s mixed strategy is to play Own Tech. with probability $y = \frac{2}{3}$ and Comp Tech. with probability $1 - y = \frac{1}{3}$.

We do the same thing for the other firm. Let us now fix Firm 1’s mixed strategy to $x \in [0,1]$. Then, we compute Firm 2’s **expected payoffs** from playing Own Tech. and Comp Tech. when Firm 1 plays the mixed strategy x:

$$\begin{aligned} u_2(x, \text{Own Tech.}) &= 0 \cdot x + 2(1-x) = 2 - 2x, \\ u_2(x, \text{Comp Tech.}) &= x + 0 \cdot (1-x) = x. \end{aligned}$$

For Firm 2 to be willing to randomize over its pure strategies we need

$$\begin{aligned} u_2(x, \text{Own Tech.}) &= u_2(x, \text{Comp Tech.}) \\ \Leftrightarrow 2 - 2x = x &\Leftrightarrow 3x = 2 \Leftrightarrow x = \frac{2}{3}. \end{aligned}$$

Same thing as before. Firm 2 is indifferent between its two strategies whenever Firm 1 plays a mixed strategy $x = \frac{2}{3}$.

Hence, there is one *mixed-strategy NE* in which both firms play Own Tech. and Comp Tech. with probabilities 2/3 and 1/3. Formally:

$$\left\{\frac{2}{3},\frac{2}{3}\right\} \text{ or } \left\{\frac{2}{3}\text{Own Tech.}+\frac{1}{3}\text{Comp Tech.}; \frac{2}{3}\text{Own Tech.}+\frac{1}{3}\text{Comp Tech.}\right\}.$$

Additional. We have found one mixed-strategy NE in which both firms randomize. Let us now consider an equilibrium in which one firm randomizes (say Firm 2) and the other one (say Firm 1) is playing a pure-strategy (or a *degenerate* mixed-strategy). Is is possible here?

Assume that Firm 2 plays a (possibly degenerate) mixed-strategy $y \in [0,1]$. Assume then that Firm 1 is choosing to always play "Own Tech." (i.e., $x = 1$). Can this be a Nash equilibrium and if yes, what is the value of y?

It is quite simple to compute. We know that Firm 1 is playing "Own Tech." for sure. Then Firm 2's expected payoff simply writes $u_2(\text{Own Tech.}, y) = 0 \cdot y + 1 \cdot (1-y) = 1-y$. Firm 2 must then choose $y \in [0,1]$ so as to maximize its utility. Clearly, the best thing to do for Firm 2 is to set $y = 0$, i.e., to assign a zero probability to "Own Tech.", or in other words to play the pure strategy "Comp Tech.". Notice that this solution is exactly one of the two pure-strategy Nash equilibria that we have found before.

What if Firm 1 chooses to play "Comp Tech." for sure? Then $u_2(\text{Comp Tech.}, y) = 2 \cdot y + 0 \cdot (1-y) = 2y$ and Firm 2 will choose $y = 1$ in order to maximize its payoff. This is exactly the other pure-strategy Nash equilibrium that we have found before.

The same argument symmetrical applies to the case in which we assume that Firm 1 is randomizing (possibly with a degenerate probability distribution) and that Firm 2 is playing a pure-strategy.

We can therefore conclude that there is no equilibrium in which one firm randomizes while the other one do not. Either they both play a pure-strategy or they both play a (nondegenerate) mixed-strategy.

5. With the superior technology, the matrix of payoffs now writes as:

		Firm 2	
		y Own Tech.	$1-y$ Comp Tech.
Firm 1	Own Tech. x	(0,0)	($\underline{3}$,$\underline{1}$)
	Comp Tech. $1-x$	($\underline{1}$,$\underline{2}$)	(0,0)

There is no change in *pure-strategy NE*. As far as *mixed-strategy NE* are concerned, let us compute **expected payoffs** for Firm 1 when 2 plays the mixed strategy y:

$$u_1(\text{Own Tech.}, y) = 0 \cdot y + 3(1-y) = 3 - 3y,$$
$$u_1(\text{Comp Tech.}, y) = 1 \cdot y + 0(1-y) = y.$$

For Firm 1 to be willing to randomize over its pure strategies we need

$$u_1(\text{Own Tech.}, y) = u_2(\text{Comp Tech.}, y)$$
$$\Leftrightarrow 3 - 3y = y \Leftrightarrow 4y = 3 \Leftrightarrow y = \frac{3}{4}.$$

In other words, we need the probability that Firm 2 plays Own Tech. to increase in order for Firm 1 to be willing to randomize over its pure strategies (the expected payoff from Own Tech. is now higher!).

Exercise 4. *Bertrand duopoly with differentiated products* (*Optional*)

Two firms compete in a market by simultaneously setting the prices, (p_i, p_j), of a differentiated good. Each firm faces a *constant marginal cost* $c = 2$. The demand for each firm's good is $q_i(p_i, p_j) = 6 - p_i + bp_j$ and $q_j(p_i, p_j) = 6 - p_j + bp_i$, where b is a parameter capturing product differentiation. Assume that $b \in [0, 1]$. Payoffs are given by each firm's profits.

1. Describe the game as a normal-form game (Players, Strategies, Payoffs).
2. Write the maximization problem for each firm.
3. Solve the maximization problem for each firm, obtaining its *reaction function*. (*Hint: Take the first derivative of profits with respect to each firm's price, taking the price set by the other as given).*
4. Find the *NE* of the game, i.e., equilibrium quantities for both firms (*Hint: Use the two reaction functions and solve for* $p_i^\star$).
5. How do prices vary with b? Explain.
6. Find the payoffs ($\pi_i(p_i^\star, p_j^\star)$) obtained by firms when they play this *NE* (*Hint: Replace equilibrium prices in profits*).

Answer of Exercise 4.

1. • Players: $N = \{\text{Firm 1}, \text{Firm 2}\}$.

- Strategies: $S_1 = S_2 = [0, \infty)$.
- Payoffs:

$$\pi_1(p_1, p_2) = (6 - p_1 + bp_2)(p_1 - 2),$$
$$\pi_2(p_1, p_2) = (6 - p_2 + bp_1)(p_2 - 2.)$$

2. When you write the maximization problem for Firm 1, remember to take the price set by Firm 2 as given.

$$\max_{p_1} \pi_1(p_1, p_2^*) = (6 - p_1 + bp_2)(p_1 - 2),$$
$$\max_{p_2} \pi_2(p_1^*, p_2) = (6 - p_2 + bp_1)(p_2 - 2).$$

3. For each firm, we compute the *first-order condition* and then solve for price.

   - Firm 1:

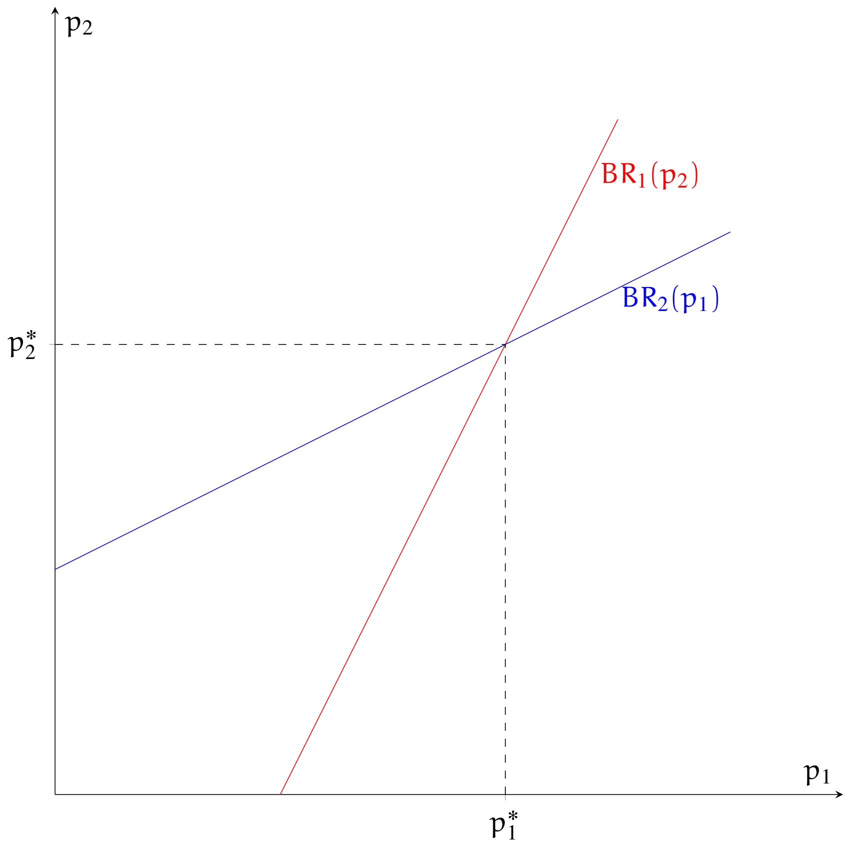
$$\frac{\partial \pi_1(p_1, p_2^*)}{\partial p_1} = 0 \implies 6 - 2p_1 + bp_2^* + 2 = 0 \implies p_1^* = \frac{8 + bp_2^*}{2}.$$

   - Firm 2:

$$\frac{\partial \pi_2(p_1^*, p_2)}{\partial p_2} = 0 \implies 6 - 2p_2 + bp_1^* + 2 = 0 \implies p_2^* = \frac{8 + bp_1^*}{2}.$$

$$\begin{cases} p_1^* = \dfrac{8 + bp_2^*}{2} \\ p_2^* = \dfrac{8 + bp_1^*}{2} \end{cases}$$

Graphically, as in the Cournot case, it is possible to plot the best-response function in the plane (p_1, p_2).



4. Solving the above linear system of two unknowns and two equations gives (simply replace q_2^* in firm 1's reaction function and solve for q_1^*):

$$p_1^* = \frac{1}{2}(8 + b(\frac{8 + bp_1^*}{2})) \Leftrightarrow p_1^* = 2(2+b) + \frac{b^2}{4}p_1^* \Leftrightarrow p_1^*(1 - \frac{b^2}{4}) = 2(2+b) \Leftrightarrow p_1^*(\frac{4-b^2}{4}) = 2(2+b)$$

Notice that $4 - b^2 = (2-b)(2+b)$ so that the above equation simplifies to

$$p_1^* = \frac{8}{2-b}.$$

Then we obtain that

$$p_2^* = \frac{8}{2-b}.$$

5. When b goes to 0, notice that the goods are no more substitutes. Therefore, each firm behaves as a monopolist and sets $p_i^* = 4$ (Recall that the monopolist sets a price of $p^m = \frac{a+c}{2}$ where here $a = 6$ and $c = 2$).

6. $\pi_1(p_1^*, p_2^*) = \pi_2(p_1^*, p_2^*) = \left(\frac{4+2b}{2-b}\right)^2$.

Exercise 5. *Cournot Duopoly: Numerical application* (*Additional*)

Two firms compete in a market by simultaneously setting the quantity of a homogeneous good to produce. Firm 1 faces a *constant marginal cost* $c_1 = 1$, while Firm 2 faces a *constant marginal cost* $c_2 = 4$. There is no *fixed cost* of production. *Inverse market demand* is $P(Q) = 25 - 3Q$, where Q is aggregate quantity. Payoffs are given by each firm's profits.

1. Describe the game as a normal-form game.
2. Write the maximization problem for each firm.
3. Find the Nash Equilibrium of the game.
4. Find the payoffs that firms obtain when they play the Nash equilibrium.
5. Suppose each firm faces the same fixed cost $F = 10$. How is the equilibrium affected?

Answer of Exercise 5.

1. **Players:** $N := \{\text{Firm } 1, \text{Firm } 2\}$.

   Actions: Quantity choice, $q_i \in [0, \infty)$.

   Payoffs: Profits, $\pi_i(q_i, q_j) := (P(q_i + q_j) - c_i)q_i = (25 - 3(q_i + q_j) - c_i)q_i$ for each $i = 1, 2$ and $j = 1, 2$ where $i \neq j$.

2. Each firm i solves $\max_{q_i}(25 - 3(q_i + q_j) - c_i)q_i$ while taking q_j as given.

3. Solving the maximization problem of firm i gives the following FOC:

$$25 - 3(q_i + q_j) - c_i - 3q_i = 0.$$

   Solving for q_i, we obtain

$$q_i^{BR}(q_j) := \frac{25 - c_i - 3q_j}{6},$$

where this last equation represents firm i's best response function. The Nash equilibrium is obtained by solving the linear system of equations formed by the two best-response function.[1]

The equilibrium quantity of firm i is given by

$$q_i^* = \frac{25 - 2c_i + c_j}{9}.$$

Replacing the marginal costs by their actual values we obtain:

$$q_1^* = 3,$$
$$q_2^* = 2.$$

4. Equilibrium payoffs are the firms' profits evaluated at equilibrium quantities. Hence

$$\begin{aligned}\pi_i^*(q_i^*, q_j^*) &= (25 - 3(q_i^* + q_j^*) - c_i)q_i^* \\ &= \frac{(25 - 2c_i + c_j)^2}{27}.\end{aligned}$$

Replacing the marginal costs by their actual values we obtain:

$$\pi_1^* = 27,$$
$$\pi_2^* = 12.$$

5. When facing the additional fixed cost F, the profit of firm i rewrites as follows:

$$\pi_i(q_i, q_j) := (P(q_i + q_j) - c_i)q_i - F = (25 - 3(q_i + q_j) - c_i)q_i - F.$$

Notice that the F does not interact with the quantity variable. Hence, neither the best response functions nor the equilibrium quantities will be affected by the fixed cost. Only the equilibrium payoffs are affected. Those are easily computed as being the previous equilibrium payoffs minus the fixed cost, namely $\pi_1^* - F = 17$ and $\pi_2^* - F = 2$.

[1] Equivalently, the Nash equilibrium is obtained by the fixed point solution to $q_i^{BR}(q_j^{BR}(q_i)) = q_i$ and $q_j^{BR}(q_i^{BR}(q_j)) = q_j$.