Practice session 4

Game Theory - MSc EEBL

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Exercise 1. *Collusion in Cournot*

Consider a *Cournot game* in which two firms, 1 and 2, produce an *homogeneous good* and interact an *infinite number of times*. Both firms have a common *discount factor* $\delta \in (0,1)$, which is a measure of their *patience* concerning future profits. In each of the (infinite and identical) stage games, the two firms simultaneously set the quantity of the goods they produce (q_1, q_2). The *marginal cost* of production is $c = 3$. Inverse market demand in each stage is given by

$$P(Q) = 9 - Q,$$

where $Q := q_1 + q_2$.

1. Find the *Nash equilibrium* quantity and profits of the *stage game*.

2. What are the maximum profits that can be achieved by the two firms if they cooperate?

   Consider now the following grim trigger strategy: "*In stage* t*, produce* $q_i = \frac{Q^m}{2}$ *if firm* j *has produced* $q_j = \frac{Q^m}{2}$ *in all previous periods; otherwise, produce the Cournot quantity forever*".

3. Consider first a subgame starting at period t such that at least one firm has played something different from half the monopoly quantity in the past. Assume that firm j plays the trigger strategy and show that firm i has no incentive to deviate from the trigger strategy.

4. Consider a subgame starting at period t such that both firms have always cooperated in the past, that is, they have chosen $q_i = q_j = \frac{Q^m}{2}$ for all periods from 0 to $t-1$.

   (a) Assume that firm j plays the trigger strategy. Find the profit of firm i when it sticks to the trigger strategy.

(b) Still assume that firm j plays the trigger strategy but now assume firm i deviates from the trigger strategy. Find q_i^d, the deviation quantity that maximizes firm i's profit when it deviates.

(c) Finally, compute the profit that firm i can obtained by deviating.

(d) Compare the two profits (trigger strategy and deviation) and deduce a condition on δ.

Answer of Exercise 1.

1. The Nash equilibrium of the stage game is simply the standard Cournot duopoly result. We can use the formulas obtained in class and we get:

$$q_1^* = q_2^* = \frac{a-c}{3} = \frac{9-3}{3} = 2,$$
$$\pi_1^C = \pi_2^C = \frac{(a-c)^2}{9} = \frac{(9-3)^2}{9} = 4.$$

2. We know that the maximum attainable profit in this environment is the monopoly profit. It corresponds to a total supplied quantity Q^m such that Q^m solves $\max_Q (P(Q)-c)Q$. In our example, we have to solve for

$$\max_Q (9-Q-3)Q \Leftrightarrow \max_Q (6-Q)Q.$$

The first-order condition of this problem writes $6-Q-Q=0$ which gives that $Q^m = 3$. Hence the monopoly profit is given by $\pi^m = (6-Q^m)Q^m = (6-3)3 = 9$.

Here, one way firms could cooperate would be that they produce each half the monopoly quantity, i.e. $q_i = q_j = \frac{Q^m}{2}$ and so they share equally the monopoly profit, that is, they each obtain $\frac{\pi^m}{2} = 9/2$.

3. If firm j plays the trigger strategy then it must produce the Cournot quantity from period t to infinity. By definition, the best response of firm i to firm j producing the Cournot quantity is also to produce the Cournot quantity. Hence, firm i best move in those subgames coincides with what the trigger strategy postulates.

4. (a) According to the trigger strategy, firm i must produce half of the monopoly quantity forever and it also expects firm j to produce half of the monopoly quantity forever. Therefore, at period t firm i expects to receive half of the monopoly profit forever. Its

discounted profit starting at period t writes:

$$\begin{aligned}\pi_i^t = \frac{9}{2} + \delta^1 \frac{9}{2} + \delta^2 \frac{9}{2} + \ldots &= \frac{9}{2} \sum_{\tau=0}^{\infty} \delta^\tau \\ &= \frac{9}{2(1-\delta)}.\end{aligned}$$

(b) When firm i considers deviating, it knows that firm j is going to produce half the monopoly quantity in period t but then it is going to play the Cournot quantity from period $t+1$ to infinity.

We now have to find what is the best possible deviation for firm i in period t. Given that firm j will produce $q_j = \frac{Q^m}{2}$ in period t firm i faces the following problem

$$\begin{aligned}\max_{q_i^d} \left(P(q_i^d + \frac{Q^m}{2}) - c\right) q_i^d &\Leftrightarrow \max_{q_i^d} (9 - q_i^d - \frac{3}{2} - 3) q_i^d \\ &\Leftrightarrow \max_{q_i^d} (\frac{9}{2} - q_i^d) q_i^d.\end{aligned}$$

The FOC of this problem writes $\frac{9}{2} - q_i^d - q_i^d = 0$ so that $q_i^d = \frac{9}{4}$. This represents the best possible deviation for firm i when the firm j is producing half the monopoly quantity.

(c) We can then compute the instantaneous profit of firm i when it deviates in period t. We have

$$\pi_i^d = (9 - \frac{9}{4} - \frac{3}{2} - 3)\frac{9}{4} = \frac{81}{16}.$$

Now we have to compute the discounted profit of firm i when it deviates. In period t, firm i will get $\pi_i^d = \frac{81}{16}$, and from period $t+1$ to infinity it will get 4 in each period (Cournot profit, see question 1). We obtain

$$\begin{aligned}\pi_i^{t,dev} &= \frac{81}{16} + 4 \sum_{\tau=1}^{\infty} \delta^\tau \\ &= \frac{81}{16} + \frac{4\delta}{1-\delta}.\end{aligned}$$

(d) Now, we only have to compare the profit when sticking to the trigger strategy and when deviating. It is better to play the trigger strategy whenever

$$\pi_i^t \geqslant \pi_i^{t,dev} \Leftrightarrow \frac{9}{2(1-\delta)} \geqslant \frac{81}{16} + \frac{4\delta}{1-\delta}.$$

Solving this inequality for the discount factor gives:

$$\delta \geqslant \frac{9}{17} \approx 0.53.$$

Exercise 2. *Repeated Prisoner's Dilemma*

Consider the following static Prisoner's dilemma.

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	(1,1)	(-1,2)
	Defect	(2,-1)	(0,0)

1. Find all Nash equilibria of the static game.

2. Assume now that the game is repeated T times, where $\mathsf{T} < \infty$. Find the subgame perfect Nash equilibrium (*Hint: as there is a finite number of repetitions, you can use backward induction*).

3. Assume now that the game is repeated an infinite number of periods. Furthermore, let $\delta \in (0, 1)$ be the discount factor common to both players.

    (a) Consider the *non-forgiving trigger strategy* for both players. What does it mean in this particular game?

    (b) Consider any subgame of the game starting at period t. Among all possible history of play, there is only two relevant cases: (i) No one has played D in the past, and (ii) someone has played D in the past. In each case, compute the payoff a player obtains when playing D and when playing C.

    (c) Using the previous payoff, find conditions on δ such that the *non-forgiving trigger strategy* is a best response in each case (i) and (ii).

    (d) When those conditions are met, what does it mean?

Answer of Exercise 2.

1. There is only one Nash equilibrium and it is a pure-strategy Nash equilibrium in which both players defect. It is easy to see that there is no mixed-strategy Nash equilibrium as "Defect" is a dominant strategy for each player, hence, it would necessarily be impossible to find indifference conditions.

2. Using backward induction, we can simply start by looking at the T-th period. Consider any possible history of play for the $(T-1)$ periods. Notice that in any case, the strategy "Defect" is a dominant strategy for each player at the last period, regardless the history of play. Therefore, players will both play the static pure-strategy Nash equilibrium at period T. Then, looking at period $(T-1)$, the same thing happens, i.e., "Defect" is again a dominant strategy for each player and so they will both choose to "Defect". Applying this reasoning for all previous period leads to the same conclusion. Therefore, the only SPNE of the finite repeated game is the one in which each player chooses to defect at every period.

3. (a) The non-forgiving trigger strategy for each player is as follows:

$$\begin{cases} \text{Play C forever unless someone has played D in the past} \\ \text{Play D forever if someone has ever played D in the past} \end{cases}$$

   This strategy is said to be *non-forgiving* as D will be played forever if any of the player tried to outsmart the other at any period, even only once.

   (b) **Case (i).** Let us first start with case (i), that is, we consider the subgame starting at period t and no one has ever played D so far. Assume player 2 sticks to the non-forgiving trigger strategy, and consider the choice of player 1. If player 1 chooses to cooperate (forever), then they can expect a payoff of

$$1 + \delta + \delta^2 + \delta^3 + \cdots = \sum_{k=0}^{\infty} \delta^k = \frac{1}{1-\delta}.$$

   If instead player 1 chooses to defect (and if they do at period t, then player 2 will D forever after so that you can convince yourself that player 1 will also have interest in defecting forever as well) they get

$$2 + \delta \cdot 0 + \delta^2 \cdot 0 + \cdots = 2$$

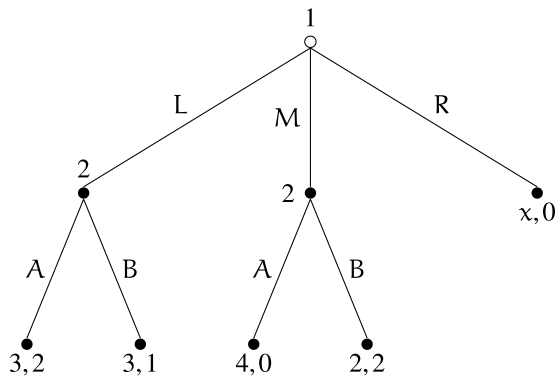
   Case (ii). Now consider case (ii), that is, the subgame starting at t in which someone has played D in the past. Again, if we assume that player 2 sticks to the non-forgiving trigger strategy it means that player 2 plays D forever. Considering player 1's choice is quite obvious. Clearly, cooperating in any period yields a payoff of -1 in that period while choosing to defect gives a payoff of 0.

(c) For C forever to be a best response in any subgame in case (i) we simply need that $\frac{1}{1-\delta} \geqslant 2$ which is equivalent to $\delta \geqslant \frac{1}{2}$. There is no condition for D to be a best response in case (ii).

(d) When the condition $\delta \geqslant \frac{1}{2}$ is satisfied, we have that the non-forgiving strategy is a best response in every subgame, that is, the non-forgiving trigger strategy yields a subgame perfect Nash equilibrium.

Exercise 3. *Simple characterization of SPNE in a dynamic game*

Consider the following **dynamic game of complete information**.



Throughout the analysis, we will consider two cases for the value of x. Either $x = 1$ or $x = 5$. It will be specified which value of x you have to consider to answer each question.

For the following questions (1, 2, 3 and 4) let $x = 1$.

1. Write the payoff matrix of this game and find all Nash equilibria.
2. What is player 2's best response after L? After M?
3. Would your answer to question 2 be different if $x = 5$? Justify.
4. Find the subgame perfect Nash equilibrium.

   Now assume that $x = 5$ for questions 5 and 6.

5. Find the subgame perfect Nash equilibrium.
6. Explain carefully why the SPNE of question 5 differs from the one you found in question 4.

7. If x can take any positive value ($x \in \mathbb{R}_+$), what is the minimal value of x such that the SPNE is that of question 5 and not that of question 4?

Answer of Exercise 3.

1. The payoff matrix of this game when $x = 1$ is as follows.

		Player 2			
		AA	AB	BA	BB
	L	$(3,2)$	$(3,2)$	$(3,1)$	$(3,1)$
Player 1	M	$(4,0)$	$(2,2)$	$(4,0)$	$(2,2)$
	R	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$

   Underline best responses in the matrix gives:

		Player 2			
		AA	AB	BA	BB
	L	$(3,\underline{2})$	$(\underline{3},\underline{2})$	$(3,1)$	$(\underline{3},1)$
Player 1	M	$(\underline{4},0)$	$(2,\underline{2})$	$(\underline{4},0)$	$(2,\underline{2})$
	R	$(1,\underline{0})$	$(1,\underline{0})$	$(1,\underline{0})$	$(1,\underline{0})$

   So that we have only one (pure-strategy) Nash equilibrium, namely, (L, AB).

2. For player 2, the best response after L is A and after M is B.

3. The answer to the previous question does not change if $x = 5$ as there is no reason that player 2 exhibits a different best response after L or M when we change something that occurs after R.

4. Here we can either find the SPNE using backward induction or simply notice that the game has only one Nash equilibrium and thus it must also be the SPNE. Hence, the SPNE is (L, AB) when $x = 1$.

5. Now assume that $x = 5$. We know from question 2 and 3 that player 2's best responses after L and M are A and B, respectively. The only change is that now, player 1 may not want to play L anymore as it might be better to play R instead given that $x = 5$ now.

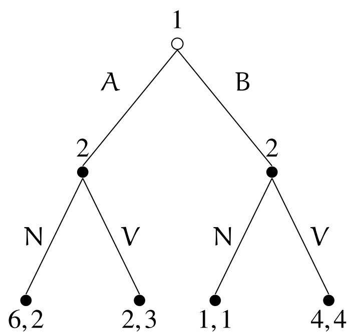
   And indeed, player 1 can either play L and get 3, play M and get 2 or play R and get 5. Player 1 will now definitely prefer to play R and thus the SPNE is (R, AB) when $x = 5$.

   Notice once again that changing the value of x has no effect on player 2's best responses after L and M but only changes the best response of player 1.

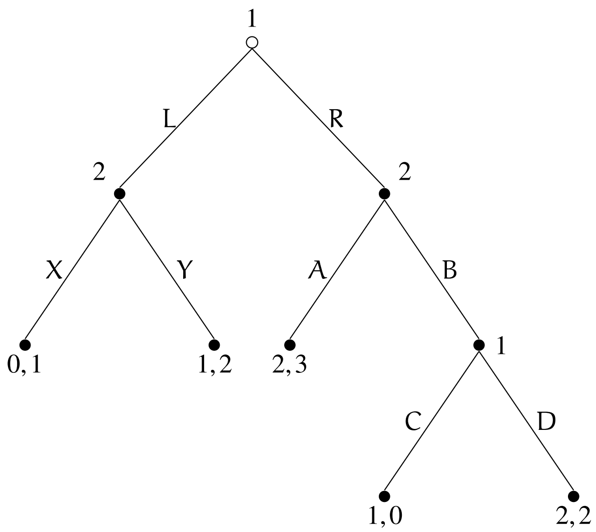
6. For player 1 to be indifferent to play L and R we must have $x = 3$. Hence we have the following characterization of SPNE: (i) for $0 \leqslant x < 3$ the SPNE is (L, AB), (ii) for $x = 3$ both (L, AB) and (R, AB) are SPNE of the game, and (iii) for $x > 3$ the SPNE is (R, AB).

Exercise 4. ***Additional dynamic games***

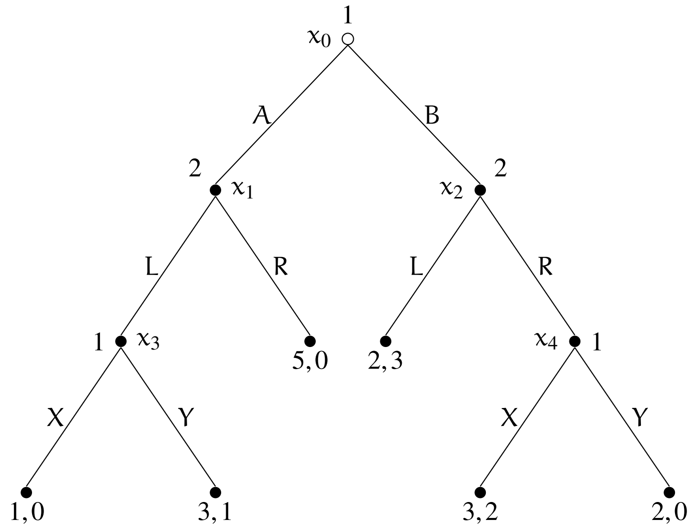
1. Find the Nash equilibria and the SPNE in the following game.



2. Find the SPNE of the following game.



3. Find the SPNE of the following game.



Answer of Exercise 4.

1. The payoff matrix writes as follows.

		Player 2			
		NN	NV	VN	VV
Player 1	A	$(6,2)$	$(6,2)$	$(2,3)$	$(2,3)$
	B	$(1,1)$	$(4,4)$	$(1,1)$	$(4,4)$

Hence we have:

		Player 2			
		NN	NV	VN	VV
Player 1	A	$(\underline{6},2)$	$(\underline{6},2)$	$(\underline{2},\underline{3})$	$(2,\underline{3})$
	B	$(1,1)$	$(4,\underline{4})$	$(1,1)$	$(\underline{4},\underline{4})$

So we have two Nash equilibria: (A, VN) and (B, VV).

But it is clear that for player 2 playing N after B is not sequentially rational ($1 < 4$). Then our only SPNE here is (B, VV).

2. Let us proceed by backward induction. First, after L player 2 wants to play Y. Second, after B player 1 wants to play D. So after R player 2 can choose to play A and obtain 3 or play B and obtain 2. Hence, player 2 wants to play A after R.

So now we can fully solve for the SPNE by finding what player 1 will want to do first. If player 1 plays L they get 1 as player 2 will play Y. If player 1 plays R, they get 2 as player 2 will play A. Player 1 will therefore prefer R to L.

Hence, the SPNE of the game is (RD, YA).

3. Using backward induction we first notice that player 1 is going to play Y at x_3 ($1 > 0$) and X at x_4 ($2 > 0$). Hence player 2 will play L at x_1 ($1 > 0$) and L at x_2 ($3 > 2$). Finally, player 1 will play A at x_0 ($3 > 2$).

   It follows that the SPNE is (AYX, LL).