Exam Statistics 29^{th} October 2018 (D)

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This is a closed book exam. Answer all the following questions and solve all the following exercises. You have two hours to complete the exam.

Exercise 1

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables. Let $f_\theta(x)$ be the density function. Let $\hat{\theta}$ be the MLE of θ, θ_0 be the true parameter, $L(\theta)$ be the likelihood function, $l(\theta)$ be the loglikelihood function, and $I(\theta)$ be the Fisher information matrix

1. Suppose $p_\theta(x)$ is the probability distribution of a Poisson with parameter θ. The Fisher information is

   (a) $I(\theta) = \frac{1}{\theta(1-\theta)}$

   (b) $I(\theta) = \frac{\theta(1-\theta)}{1}$

   (c) $I(\theta) = \frac{1}{\theta^2}$

   (d) $I(\theta) = \theta^2$

   (e) $I(\theta) = \theta$

   (f) $I(\theta) = \frac{1}{\theta}$

2. Under which of the following case, $\hat{\theta}$ does not approximately follow a normal distribution?

   (a) $f_\theta(x)$ is the probability distribution function of a Poisson with positive mean $exp(\theta)$

   (b) $f_\theta(x) = \theta x^{\theta-1}$ for $x \in (0,1)$is with $\theta > 0$.

   (c) $f_\theta(x)$ is the density distribution function of a $Pareto(\theta, x_m)$, where x_m is known, but θ unknown with $\theta > 0$

   (d) $f_\theta(x)$ is the density distribution function of a Uniform distribution $U(0, 2\theta)$, with $\theta > 0$

3. Since β is the probability of Type II error then $1-\beta$ is:
   (a) Probability of rejecting H_0 when H_0 is true
   (b) Probability of NOT rejecting H_0 when H_0 is true
   (c) Probability of NOT rejecting H_0 when H_1 is true
   (d) Probability of rejecting H_0 when H_1 is true
   (e) $1-\alpha$

Exercise 2

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = (1+\theta)(1-x)^{\theta} I_{(0,1)}(x) \qquad \theta > -1$$

1. Find a sufficient statistics for θ
2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and compare with $\hat{\theta}_{MOM}$ method of moments estimator (MOM).
3. Find maximum likelihood estimator for the population mean.
4. Compute the score function and the Fisher information.
5. Specify asymptotic distribution of $\hat{\theta}_{MLE}$.
6. Any guess about asymptotic distribution of $\hat{\theta}_{MOM}$?
7. Suppose form a random sample of 300 random variables you obtain $\sum_{i=1}^{300} log\,(1-x_i) = -100$. Complete ALL of the following questions:
   (a) Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 3$ versus $H_1 : \theta \neq 3$, specify the asymptotic distribution and verify the null hypothesis.
   (b) Find the Wald test statistic for testing $H_0 : \theta = 3$ versus $H_1 : \theta \neq 3$, specify the distribution and verify the null hypothesis.
   (c) Find the Score test statistic for testing $H_0 : \theta = 3$ versus $H_1 : \theta \neq 3$, specify the distribution and verify the null hypothesis.

Exercise 3

Let $(X_1, \ldots, X_n)$ be n random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$
Is the following estimator an unbiased estimator for μ^2

$$T = \frac{n}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2 - \sum_{j=1}^{n} X_j^2 + X_1 \sum_{j=2}^{n} X_j$$

Exercise 4

Imagine you are rolling a die with an unknown number of faces, indicated with parameter θ. You are rolling the die 5 times and observing the following faces:

$$10 \qquad 10 \qquad 3 \qquad 7 \qquad 5$$

Provide method of moment and maximum likelihood estimate of the unknown parameter θ.

Exercise 5

Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from the following distribution

$$\frac{\theta_0(-log\theta_0)^x}{x!}$$

Find maximum likelihood estimator for θ_0.

Exercise 6

Let $(X_1, \ldots, X_n)$ be a random sample of size n from the uniform continuous distribution $U(0, \theta)$. Is the random variable $\frac{max(X_1, X_2, \ldots, X_n)}{\theta}$ a pivotal quantity for θ? Provide the definition of a pivotal quantity for a parameter θ.

Exercise 7

Provide correct statement for Cramer-Rao inequality