Exam Statistics 24^{th} October 2022 (a)

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Exercise 1

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables. Let $f_\theta(x)$ be the density function. Let $\hat{\theta}$ be the MLE of θ, θ_0 be the true parameter, $L(\theta)$ be the likelihood function, $l(\theta)$ be the loglikelihood function, and $I(\theta)$ be the Fisher information matrix

1. The *Invariance* property of MLE's implies that:

   a Their variance approaches zero as the sample size increases without limit

   b Their variance achieves the Cramer-Rao lower bound

   c Any continuous function of an MLE is the MLE for that function of the parameter(s)

   d Any monotonic function of an MLE is the MLE for that function of the parameter(s)

2. When estimating a scalar parameter, the Cramer-Rao lower bound is:

   a The smallest variance that that any MLE can achieve

   b A hypothetical, but not always attainable, minimum value for the variance of any estimator for this parameter

   c The bias that can be achieved by any estimator for this parameter

   d The value of Fisher's "Information Measure"

3. The MLE estimator for the median of the Gaussian distribution with parameter (μ, σ^2) is

   a $Median(X_1, \ldots, X_n)$

   b $\bar{X}$

   c $\bar{X} \times log(2)$

   d $\frac{\sum log X_i}{n}$

4. Which of the following statements is TRUE?

a A very low significance level increases the chances of a Type I error.

b Type I error occurs when the null hypothesis is rejected and the null hypothesis is actually true.

c Type I error occurs when the null hypothesis is accepted and the null hypothesis is actually true.

d Type II error occurs when the null hypothesis is rejected and the null hypothesis is actually false.

Exercise 2

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as follows:

$$f(x;\theta) = \frac{\theta^3}{2} x^2 exp(-\theta x) \qquad x > 0$$

1. Find sufficient statistics for θ
2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.
3. Compute the score function and the Fisher information.
4. Specify asymptotic distribution of $\hat{\theta}_{MLE}$.
5. Find $\hat{\theta}_{MOM}$ method of moments estimator (MOM) for θ and specify asymptotic distribution
6. Is the following estimator an unbiased estimator of θ^2?

$$T_(X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{2}{X_i^2}$$

7. Which of the following estimator would you prefer to estimate the population variance

$$\begin{aligned} T_1(X_1, \ldots, X_n) &= \frac{\bar{X}^2}{3} \\ T_2(X_1, \ldots, X_n) &= \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1} \end{aligned}$$

8. Suppose from a random sample of size $n = 200$ you observe the following sample value $\sum x_i = 157.895$, is it possible to provide a reasonable estimate for θ. Justify your answer.
9. Let $\alpha = 0.05$, complete the following questions:

(a) Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 4$ versus $H_1 : \theta \neq 4$, specify the distribution and verify the null hypothesis.

(b) Find the Wald test statistic for testing $H_0 : \theta = 4$ versus $H_1 : \theta \neq 4$, specify the distribution and verify the null hypothesis.

(c) Find the Score test statistic for testing $H_0 : \theta = 4$ versus $H_1 : \theta \neq 4$, specify the distribution and verify the null hypothesis.

HINT: Recall of the Gamma Distribution

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad x > 0$$

and $E(X) = \frac{\alpha}{\beta}$ and $Var(X) = \frac{\alpha}{\beta^2}$

Exercise 3

Consider a population X a random variable that is equal to 1 with probability θ^2, X is equal to 2 with probability $2\theta(1-\theta)$ X is equal to 3 with probability $(1-\theta)^2$ where $0 < \theta < 1$. And let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables

1. Show that following estimator

$$T = \frac{\sum_i I(X_i = 1)}{n} + \frac{\sum_i I(X_i = 2)}{2n}$$

is an unbiased estimator of θ

2. Find $\hat{\theta}_{MOM}$

3. Find $\hat{\theta}_{MLE}$

Exercise 4

Let X be a Poisson random variable, with parameter λ such that:

$$P(x|\lambda) = \frac{\lambda^x exp(-\lambda)}{x!}$$

Propose and discuss an approximate confidence interval for parameter λ.

Exercise 5

Provide correct statement for Cramér-Rao inequality