Exam Statistics 5^{th} February 2024 (b)

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This is a closed book exam. Answer all the following questions and solve all the following exercises. You have one hour and half to complete the exam.

Exercise 1

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables. Let $f_\theta(x)$ and $F_\theta(x)$ be the density function and cumulative distribution function respectively. Let $\hat{\theta}$ be the MLE of θ, θ_0 be the true parameter, $L(\theta)$ be the likelihood function, $logL(\theta)$ be the loglikelihood function, and $I(\theta)$ be the Fisher information matrix

1. A statistic $T = T(X)$ is a sufficient statistic for θ if
   (a) The distribution of the sample conditionally to T does not depend on θ
   (b) Its distribution does not depend on θ
   (c) It can be computed without knowing the value of θ
   (d) Its distribution depends on θ
2. For an estimator to be consistent, the unbiasedness of the estimator is a condition:
   (a) Necessary
   (b) Sufficient
   (c) Necessary and Sufficient
   (d) Neither Necessary nor Sufficient

Exercise 2

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as follows:

$$f(x;\theta) = \frac{1}{\theta} exp\left\{-\frac{1}{\theta}(x-\mu)\right\} \qquad x > \mu, \quad \theta > 0$$

1. Find a sufficient statistics for the three following families of distributions:
   (a) θ is known
   (b) μ is known
   (c) Both the parameters (μ, θ) are unknown,

Assume μ is known and equal to 3, Then, answer the followig questions:

2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.
3. Find $\hat{\theta}_{MOM}$ method of moment estimator (MOM) for θ.
4. Compute the score function and the Fisher information.
5. Specify asymptotic distribution of $\hat{\theta}_{MLE}$.
6. Suppose form a random sample of 250 random variables you obtain $\sum_{i=1}^{250} x_i = 1200$. Let $\alpha = 0.05$, complete the following questions
   (a) Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 4$ versus $H_1 : \theta \neq 4$, specify the distribution and verify the null hypothesis.
   (b) Find the Wald test statistic for testing $H_0 : \theta = 4$ versus $H_1 : \theta \neq 4$, specify the distribution and verify the null hypothesis.
   (c) Find the Score test statistic for testing $H_0 : \theta = 4$ versus $H_1 : \theta \neq 4$, specify the distribution and verify the null hypothesis.

Exercise 3

Suppose that a random variable X follows a discrete distribution, which is determined by a parameter θ which can take only two values, $\theta = 0$ or $\theta = 1$. The parameter θ is unknown. If $\theta = 0$, then X follows a Poisson distribution with parameter $\lambda = 1$. If $\theta = 1$, then X follows a Geometric distribution with parameter $p = 0.2$. Now suppose we observe $X = 1$. Based on this data, what is the maximum likelihood estimate of θ?

Exercise 4

The daily precipitation (in mm) recorded in Vancouver area can be described by an distribution with parameter λ as;

$$f(x) = \lambda exp(-\lambda x) \qquad for\ x > 0$$

Immagine in the last year, we records only the number of days total precipitation greater than 25mm. In particular we observe 55 days of precipitation greater than 20mm un to 365 observed days. It is possible to provide a MLE estimator and the corresponding estimate for λ?

Exercise 5

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and known variance σ^2. Assume that we are interested in estimating the variance of the distribution: $\theta = \sigma^2$. Consider the following estimators and indicate for each of them if they are unbiased, asymptotic unbiased and consistent estimator or θ and motivare your ansers:

1. $$T_1 = \frac{(X_1 - X_2)^2}{2}$$

2. $$T_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

3. $$T_3 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

4. $$T_4 = \frac{1}{n}\sum_{i=1}^{n}X_i^2$$

Estimators	Unbiased	Asymptotic Unbiased	Consistent
T_1	☐	☐	☐
T_2	☐	☐	☐
T_3	☐	☐	☐
T_4	☐	☐	☐

- REMEMBER distributions:
  - Poisson: $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$
  - Geometric: $P(X = x) = p\ (1-p)^{x-1}$

Exercise 6

Provide correct statement for Neyman Pearson Lemma