Exercise

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta(1-x)^{\theta-1} I_{(0,1)}(x) \qquad \theta > 1$$

1. Find a sufficient statistics for θ
2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.
3. Compute the score function and the Fisher information.
4. Specify asymptotic distribution of $\hat{\theta}_{MLE}$.

Exercise

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta(1-x)^{\theta-1} I_{(0,1)}(x) \qquad \theta > 1$$

1. Suppose form a random sample of 200 random variables you obtain $\sum_{i=1}^{200} log\,(1-x_i) = -110$. Complete ALL of the following questions:
   1. Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the asymptotic distribution and verify the null hypothesis.
   2. Find the Wald test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the distribution and verify the null hypothesis.
   3. Find the Score test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the distribution and verify the null hypothesis.

Solution 1

$$f(x_1, \ldots, x_n) = \prod_i \theta(1 - x_i)^{\theta - 1}$$

$$f(x_1, \ldots, x_n) = \theta^n \left(\prod_i (1 - x_i)^{\theta - 1} \right)$$

Sufficient statistics

$\prod_i (1 - x_i)$

Solution 2

$$
\begin{aligned}
L(\theta|x_1,\ldots,x_n) &= \theta^n \left(\prod_i (1-x_i)^{\theta-1}\right) \\
logL(\theta|x_1,\ldots,x_n) &= nlog(\theta) + (\theta-1)\left(\sum_i log(1-x_i)\right) \\
\frac{\partial logL(\theta|x_1,\ldots,x_n)}{\partial\theta} &= \frac{n}{\theta} + \left(\sum_i log(1-x_i)\right) \\
\frac{\partial^2 logL(\theta|x_1,\ldots,x_n)}{\partial^2\theta} &= -\frac{n}{\theta^2} < 0 \qquad \forall\theta
\end{aligned}
$$

Solution 2

Maximum Likelihood Estimator

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_i log(1 - X_i)}$$

Maximum Likelihood Estimator

1.82

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Solution 3

Score Function

$$U(\theta) = \frac{n}{\theta} + \left(\sum_i log(1 - x_i) \right)$$

Fisher Information

$$I(\theta) = \frac{n}{\theta^2}$$

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Specify asymptotic distribution of $\hat{\theta}_{MLE}$

$$\hat{\theta}_{MLE} \approx N\left(\theta_0, \frac{\theta_0^2}{n}\right)$$

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Specify asymptotic distribution of $\hat{\theta}_{MLE}$

$$\hat{\theta}_{MLE} \approx N\left(\theta_0, \frac{\theta_0^2}{n} \right)$$

Wald Test

$$\hat{\theta} \sim N\left(\theta_0, \frac{1}{nI(\theta_0)}\right)$$

$$\left(\hat{\theta} - \theta_0\right)^2 \times nI(\theta_0) \sim \chi_1$$

Wald test Statistics

$$\left(\hat{\theta} - \theta_0\right)^2 \times nI(\hat{\theta})$$

Wald test Statistics

$$\left(\hat{\theta} - \theta_0\right) \times \sqrt{nI(\hat{\theta})}$$

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$$\frac{-\frac{n}{\sum_{i=1}^{n} log(1-x_i)} - \theta_0}{\frac{-\frac{n}{\sum_{i=1}^{n} log(1-x_i)}}{\sqrt{n}}}$$

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Distribution of test statistics

$$N(0,1)$$

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Observed value of test statistics

$$\frac{1.82-2}{\sqrt{0.0166}} = -1.4 \qquad p - value = 0.16$$

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Score Test

$$\frac{\left(\frac{\partial logL(\theta_0)}{\partial\theta}\right)^2}{nI(\theta)_0} \sim \chi^2_1$$

Score Test Statistics

$$\frac{\left(\frac{n}{\theta_0} + \sum_{i=1}^{n} log(1 - x_i)\right)^2}{\frac{n}{\theta_0^2}}$$

Score Test

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Distribution of the Score Test Statistics

$$N(0,1)$$

Observed value of test statistics

$$\begin{aligned} \frac{\delta logL(\theta_0}{\delta\alpha} &= \frac{200}{2} - 110 = -10 \\ \frac{logL\prime(\theta_0)}{\sqrt{nI(\theta_0)}} &= \frac{-10}{\sqrt{50}} = -1.41 \qquad pvalue = 0.16 \end{aligned}$$

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score=-1.41, p-value=0.16

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Likelihood ratio Test

Log Likelihood

$$logL(\alpha) \quad = \quad nlog\theta + (\theta - 1)\sum_{i=1}^{n} log(1 - x_i)$$

Likelihood ratio Test

$$2 \times (logL(\hat{\theta}) - logL(\theta_0)) \quad \sim \quad \chi^2_1$$

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Likelihood ratio Test Statistics

$$2 \times (logL(\hat{\theta}) - logL(\theta_0))$$

$$2 \times \left(nlog\frac{\hat{\theta}}{\theta_0} + (\hat{\theta} - \theta_0)\sum_{i=1}^{n} log(1 - x_i) \right)$$

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Distribution of the Test Statistics

$$\chi_1^2$$

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Distribution of the Test Statistics

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Observed Test Statistics

$$2 \times (logL(\hat{\theta}) - logL(\theta_0)) \quad = \quad 0.94 \qquad p - value = 0.66$$

Likelihood ratio Test

Likelihood ratio Test Statistics

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