Uniform Distribution

Example: Let $X_1, \dots, X_n$ be iid r.v. distributed as continuous uniform distribution on $[0, \theta]$. The probability distribution function of X_i for each i is:

$$f(x|\theta) = \begin{cases} \theta^{-1}, & 0 \leq x \leq \theta \\ 0, & otherwise \end{cases}$$

Consider $T = max_i X_i$ an estimator of θ, is T an unbiased estimator for θ?

Uniform Distribution

Is $T = max_i X_i$ an unbiased estimator of θ?

We need to compute $E(T)$, thus we need distribution for T

Let F be the cumulative distribution function for T,
$F_T(y) = P(T \leq y) = P(max_i X_i \leq y)$

$$P(max_i X_i \leq y) = P(X_1 \leq y, \ldots, X_n \leq y) = P(X \leq y)^n = \left(\frac{y}{\theta}\right)^n$$

$$f_T(t) = \frac{\partial F_T(y)}{\partial y} = n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}$$

$$E(T) = \int_0^\theta y f_T(y) dy = \int_0^\theta n\left(\frac{y}{\theta}\right)^n dy = \frac{n}{n+1}\theta$$

$$Bias(T) = E(T) - \theta = \frac{-1}{n+1}\theta$$

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Uniform Distribution

Is $T = max_i X_i$ a consistent estimator of θ?

$$
\begin{aligned}
f_T(t) &= n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} \\
E(T^2) &= \int y^2 f_T(y)dy = \int n\frac{y^{(n+1)}}{\theta^n}dy = \frac{n}{n+2}\theta^2 \\
Var(T) &= E(T^2) - (E(T))^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 \\
Var(T) &= \frac{n}{(n+2)(n+1)^2}\theta^2
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Uniform Distribution

Is $T_n = max_i X_i$ a consistent estimator of θ?

$$\begin{aligned}
Bias(T) &= E(T) - \theta = \frac{-1}{n+1}\theta \\
Var(T) &= \frac{n}{(n+2)(n+1)^2}\theta^2 \\
MSE(T) &= \left(\frac{n}{(n+2)(n+1)^2}\right)\theta^2 + \left(\frac{-1}{n+1}\right)^2\theta^2 \\
MSE(T) &= \left(\frac{2\theta^2}{(n+2)(n+1)}\right) \\
lim_{n\to\infty} MSE(T_m) &= 0
\end{aligned}$$

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Uniform Distribution

Two estimators for θ

- $\hat{\theta}_{MOM}$
- $\hat{\theta}_{MLE}$

Which estimator is *the best* one?

Uniform Distribution

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- $\hat{\theta}_{MOM}$
- $\hat{\theta}_{MLE}$

Which estimator is *the best* one?

Uniform Distribution: $\hat{\theta}_{MOM}$

$$E(X) = \int xf(x|\theta)dx = \int_0^\theta \frac{x}{\theta}dx = |_0^\theta \frac{x^2}{2\theta} = \frac{\theta}{2}$$

$$\bar{X} = \frac{\theta}{2}$$

$$\hat{\theta}_{MOM} = 2\bar{X}$$

Uniform Distribution: $\hat{\theta}_{MOM}$

Is $\hat{\theta}_{MOM}$ an unbiased and a consistent estimator of θ

$$
\begin{aligned}
E\left(\hat{\theta}_{MOM}\right) &= \theta \\
Var\left(\hat{\theta}_{MOM}\right) &= 4Var\left(\bar{X}\right) = 4\frac{\theta^2}{12n} = \frac{\theta^2}{3n}
\end{aligned}
$$

$$
\begin{aligned}
MSE\left(\hat{\theta}_{MOM}\right) &= \frac{\theta^2}{3n} \\
lim_{n\rightarrow} MSE(\hat{\theta}_{MLE}) &= 0
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Comparison between two estimators: $\hat{\theta}_{MLE}$ and $\hat{\theta}_{MOM}$

Which estimator is *the best* one?

$$
\begin{aligned}
MSE\left(\hat{\theta}_{MLE}\right) &\leq Var\left(\hat{\theta}_{MOM}\right) \\
\frac{2}{(n+1)(n+2)}\theta^2 &\leq \frac{\theta^2}{3n} \\
6n &\leq n^2 + 3n + 2 \\
n^2 - 3n + 2 &\geq 0 \\
n &> 2
\end{aligned}
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Exercise

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta^2 x \exp(-\theta x) \qquad x > 0$$

Is $T(X_1, \ldots, X_n) = 1/X_1$ an unbiased estimator of θ?

Exercise

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta^2 x \exp(-\theta x) \qquad x > 0$$

$$\begin{aligned}
E(T(X_1, \ldots, X_n)) &= \int_{(x1, x2, \ldots, x_n)} \frac{1}{x_1} f(x1, x2, \ldots, x_n) dx_1 dx_2 \ldots dx_n \\
E(T(X_1, \ldots, X_n)) &= \int_0^\infty \frac{1}{x_1} \theta^2 x_1 \exp(-\theta x_1) dx_1 \\
E\left(\frac{1}{X_1}\right) &= \theta
\end{aligned}$$

Exercise

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with $E(X) = \mu$, $Var(X) = \sigma^2$, Are the following estimators unbiased estimator for σ^2?

$$
\begin{aligned}
T_1(X_1, \ldots, X_n) &= \frac{(X_1 - X_2)^2}{2} \\
T_2(X_1, \ldots, X_n) &= \frac{(X_1 + X_2)^2}{2} - X_1 X_2
\end{aligned}
$$

Exercise

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables with expected value μ and variance σ^2. Consider the following estimator of μ:

$$T_n(a) = a \times X_n + (1-a) \times \bar{X}_{n-1}$$

where X_n is the $n-th$ observed random variable and $\bar{X}_{n-1}$ is the sample mean based on $n-1$ observations.

1. Find value of a such that $T_n(a)$ is an unbiased estimator for μ
2. Find value of $a\star$ such that $T_n(a\star)$ is the most efficient estimator for μ within the class $T_n(a)$?
3. Define concept of efficiency

Poisson distribution

Example:

Let X be distributed as a Possion;

$$f(x;\lambda) = \frac{\lambda^x exp(-\lambda)}{x!}$$

Compare the following estimator for $\exp(-\lambda)$:

$$T_1 = exp\left(-\bar{X}\right) \qquad T_2 = \frac{\sum_{i=1}^{n} I(X_i = 0)}{n}$$

Poisson distribution