

Testing Hypothesis

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Hypothesis Testing

"It is a mistake to confound strangeness with mystery"

Sherlock Holmes
A Study in Scarlet

Outline

1 Large Sample Tests

- Wald Test
- Score Test
- Likelihood Ratio Test

2 p-value

- Application to Poisson Distribution

Large Sample Tests

Why do we need large sample hypothesis testing?

The advantage of a large sample for testing is that we do not need the distribution of the test statistics: less computations needed! ... but results hold asymptotically

Large Sample Tests

Suppose we wish to test $H_0 : \theta = \theta_0$ against $H_0 : \theta \neq \theta_0$. The likelihood-based give rise to several possible tests

Large Sample Tests

Let $\log L(\theta)$ denote the loglikelihood and $\hat{\theta}_n$ the consistent root of the likelihood equation. Intuitively, the farther θ_0 is from $\hat{\theta}_n$, the stronger the evidence against the null hypothesis.

But how far is far enough?

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But how far is far enough?

If θ_0 is close to $\hat{\theta}_n$

- $\log L(\theta_0)$ should also be close to $\log L(\hat{\theta}_n)$
- $\log L'(\theta_0)$ should also be close to $\log L'(\hat{\theta}_n) = 0$.

Large Sample Tests

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Large Sample Tests

Wald test

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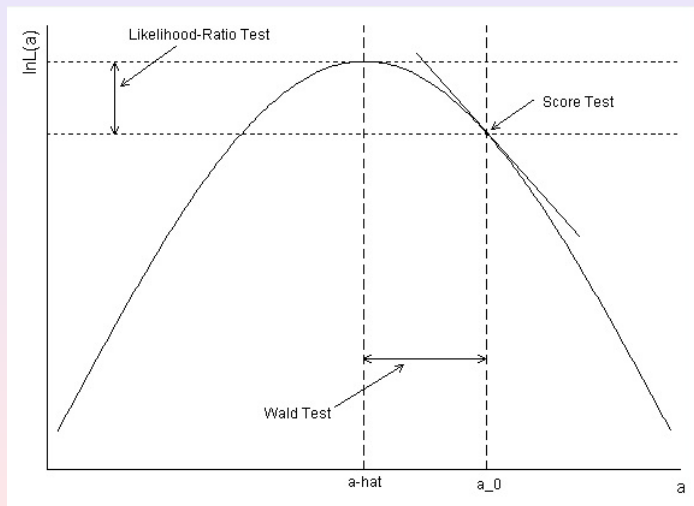
Likelihood ratio test

If we base a test upon the value of $\log L(\hat{\theta}_n) - \log L(\theta_0)$

(Rao) score test

If we base a test upon the value of $\log L'(\theta_0)$

Three asymptotic tests



Large Sample Tests

We argued that under certain regularity conditions, the following facts are true under H_0 :

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \longrightarrow N\left(0, \frac{1}{I(\theta_0)}\right)$$

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$$-\frac{1}{n}(\log L''(\theta_0)) \longrightarrow I(\theta_0)$$

Wald Test

Under certain regularity conditions, the maximum likelihood estimator $\hat{\theta}$ has approximately in large samples a (multivariate) normal distribution with mean equal to the true parameter value and variance-covariance matrix given by the inverse of the information matrix, so that

$$\hat{\theta} \sim N\left(\theta_0, \frac{1}{nI(\theta_0)}\right)$$

Wald Test

Wald test statistics (parameter θ scalar)



$$(\hat{\theta} - \theta_0) \times \sqrt{nl(\theta_0)} \sim N(0, 1)$$

- A consistent estimator is used

$$(\hat{\theta} - \theta_0) \times \sqrt{nl(\hat{\theta})} \sim N(0, 1)$$

- More generally, often it is written as:

$$\frac{\hat{\theta} - \theta_0}{\sqrt{\text{Var}(\hat{\theta})}} \sim N(0, 1)$$

Wald Test Statistics

When θ is a parameter of dimension p . This result provides a basis for constructing tests of hypotheses and confidence regions. For example under the hypothesis $H_0 : \theta = \theta_0$ for a fixed value θ_0 , the quadratic form

$$W = (\hat{\theta} - \theta_0)' I_n(\hat{\theta})(\hat{\theta} - \theta_0)$$

written also as:

$$W = (\hat{\theta} - \theta_0)' \text{var}^{-1}(\hat{\theta})(\hat{\theta} - \theta_0)$$

has approximately in large samples a chi-squared distribution with p degrees of freedom.

Comments on Wald Test Statistics

You may find also this formula:

$$W = (\hat{\theta} - \theta_0)' I_n(\theta_0) (\hat{\theta} - \theta_0)$$

and assumed an approximately chi-squared distribution with p degrees of freedom.

REMEMBER:

$$\lim_{n \rightarrow \infty} I_n(\hat{\theta}) = I_n(\theta_0)$$

Score Test

Under some regularity conditions the score itself has an asymptotic normal distribution with mean 0 and variance equal to the information matrix, so that

$$u(\theta) \sim N(0, I_n(\theta))$$

Score Test Statistics

$$u(\theta_0) \sim N(0, I_n(\theta_0))$$

$$\frac{\frac{\partial \log L(\theta | x_1, \dots, x_n)}{\partial \theta}}{\sqrt{nI(\theta_0)}} \sim N(0, 1)$$

Score Test

When θ is a parameter of dimension p . Under some regularity conditions the score itself has an asymptotic normal distribution with mean 0 and variance-covariance matrix equal to the information matrix, so that

$$u(\theta) \sim N_p(0, I_n(\theta))$$

Score Test

This result provides another basis for constructing tests of hypotheses and confidence regions. For example under $H_0 : \theta = \theta_0$ for a fixed value θ_0 , the quadratic form

$$Q = u(\theta_0)' I_n^{-1}(\theta_0) u(\theta_0)$$

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has approximately in large samples a chi-squared distribution with p degrees of freedom.

Likelihood Ratio Tests

Under certain regularity conditions, minus twice the log of the likelihood ratio has approximately in large samples a chi-square distribution

$$-2\log\lambda = 2\log L(\hat{\theta}, x) - 2\log L(\theta_0, x)$$

In case θ is a scalar, $-2\log\lambda$ has a chi-square distribution with 1 degree of freedom.

Likelihood Ratio Tests

Under certain regularity conditions, minus twice the log of the likelihood ratio has approximately in large samples a chi-square distribution with degrees of freedom equal to the difference in the number of parameters between the two models. Thus,

$$-2\log\lambda = 2\log L(\hat{\theta}_{\omega_2}, x) - 2\log L(\hat{\theta}_{\omega_1}, x)$$

where the degrees of freedom are $\nu = \dim(\omega_2) - \dim(\omega_1)$, the number of parameters in the larger model ω_2 minus the number of parameters in the smaller model ω_1 .

Large Sample Tests

The LR, Wald, and Score tests (the "trinity" of test statistics) require different models to be estimated.

- LR test requires both the restricted and unrestricted models to be estimated
- Wald test requires only the unrestricted model to be estimated.
- Score test requires only the restricted model to be estimated.

Applicability of each test then depends on the nature of the hypotheses. For $H_0 : \theta = \theta_0$, the restricted model is trivial to estimate, and so LR or Score test might be preferred. For $H_0 : \theta > \theta_0$, restricted model is a constrained maximization problem, so Wald test might be preferred.

Large Sample Tests

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The score test and likelihood ratio test are invariant under reparameterization, whereas the Wald test is not.

Large Sample Tests: $H_0 : \theta = \theta_0$

Wald Test

Under the null, the normalised distance between $\hat{\theta}$ and θ_0 should be small.

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Lagrange multiplier (or Score) test

Under the null, the score $u(\theta_0)$ evaluated at θ_0 should be close to zero.

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Lagrange multiplier (or Score) test

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Likelihood ratio test

The ratio between the likelihood $L(x, \hat{\theta})$ evaluated at at the estimate $\hat{\theta}$ and the likelihood $L(x, \theta_0)$ evaluated at θ_0 should be close to 1

Relationship between the three tests

From the Handbook Chapter 13 by Robert Engle

These three general principles have a certain symmetry which has revolutionized the teaching of hypothesis tests and the development of new procedures:

- Essentially, the Lagrange Multiplier approach starts at the null and asks whether movement toward the alternative would be an improvement
- The Wald approach starts at the alternative and considers movement toward the null
- The Likelihood ratio method compares the two hypothesis directly on an equal basis.

p-value

A common complaint about Hypothesis Testing is the choice of the significance level α , that is essentially arbitrary. To make matters worse, the conclusions (Reject or Don't Reject H_0) depend on what value of α is used.

p-value

Definition: A p-value $p(x)$ is a test statistic satisfying $0 \leq p(x) \leq 1$ for every sample point x . Small values of $p(x)$ give evidence that H_1 is true.

Theorem: Let $W(X)$ be a test statistic such that large value of W give evidence that H_1 is true. For each sample point x , define

$$p(x) = \sup_{\theta \in \Theta_0} P_{\theta}(W(X) \geq W(x))$$

then $p(X)$ is a valid p-value.

Steps to find p-value

- 1 Let T be the test statistic.
- 2 Compute the value of T using the sample $x_1, \dots, x_n$. Say it is a .
- 3 The p-value is given by

$$p - \text{value} = \begin{array}{ll} P(T < a | H_0), & \text{if lower tail test} \\ P(T > a | H_0), & \text{if upper tail test} \\ P(|T| > |a| | H_0), & \text{if two tail test} \end{array}$$

p-value

- The smallest level at which H_0 would be rejected.
- The the probability that you would obtain evidence against H_0 which is at least as strong as that actually observed, if H_0 were true.
- The smaller the $p - value$, the stronger the evidence that H_0 is false.

Example: psychiatrist problem

A psychiatrist wants to determine if a small amount of coke decreases the reaction time in adults.

The reaction time for a specified test is assumed normally distributed and has mean equal to $\mu = 0.15$ seconds and $\sigma = 0.2$.

A sample of 81 people were given a small amount of coke and their reaction time averaged 0.11 seconds.

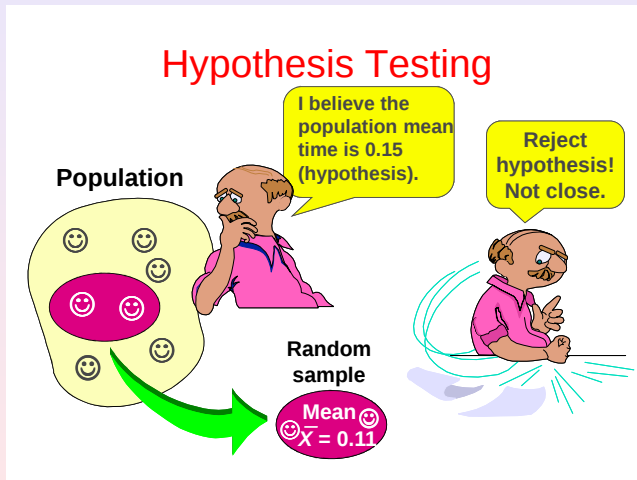
Does this sample result support the psychiatrist's hypothesis?

Example: psychiatrist problem

- The reaction time is expected to decrease
- What are null and alternative hypothesis?
- $H_0 : \mu = 0.15$ $H_1 : \mu < 0.15$
- If the null hypothesis were true, could I observe a sample mean of 0.11 from a sampling distribution centered in 0.15 and standard deviation $\sigma = 0.2$?
-

$$P(\bar{X} < 0.11 | \mu = 0.15) = P\left(\frac{\bar{X} - 0.15}{\frac{0.2}{\sqrt{81}}} < \frac{0.11 - 0.15}{\frac{0.2}{\sqrt{81}}}\right) = 0.0359$$

Example: psychiatrist problem



Example: psychiatrist problem

Hypothesis Testing
Process

I believe the
population
reaction time
is 0.15
(Hypothesis)



Population

Is $\bar{x} = 0.11 < \mu = 0.15$? The Sample
Mean Is 0.11



Sample

yes

REJECT



Hypothesis

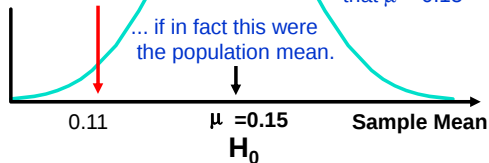
no

ACCEPT

Example: psychiatrist problem

Sampling Distribution

It is unlikely
that we would
get a sample
mean of this
value ...



p-value: Example: psychiatrist problem

- 0.0359 represents the probability to observe a sample mean with a more extreme value than 0.11.
- Probability of obtaining a test statistic as extreme as, or more extreme, ($\leq$ or $\geq$) than the actual sample value, given H_0 is true
- If we think the p-value is too low to believe the observed test statistic is obtained by chance only, then we would reject chance (reject the null hypothesis).
- Otherwise, we fail to reject chance and do not reject the null hypothesis.

Rejection Region: Example: psychiatrist problem

$$R = \left\{ (x_1, x_2, \dots, x_n) : \frac{\bar{x} - \mu_0}{\sqrt{\frac{\sigma^2}{n}}} < -z_{1-\alpha} \right\}$$

Rejection Region: Example: psychiatrist problem

- $\alpha = 0.05$

$$R = \left\{ (x_1, x_2, \dots, x_n) : \frac{\bar{x} - 0.15}{\sqrt{\frac{0.2^2}{81}}} < -1.645 \right\}$$

$-1.8 < -1.645$ Reject null hypothesis

-

$$R = \left\{ (x_1, x_2, \dots, x_n) : \frac{\bar{x} - 0.15}{\sqrt{\frac{0.2^2}{81}}} < -2.32 \right\}$$

$-1.8 < -2.32$ Accept null hypothesis

Rejection Region: Example: psychiatrist problem

$$R = \left\{ (x_1, x_2, \dots, x_n) : \bar{X} < \mu_0 - z_{1-\alpha} \sqrt{\frac{\sigma^2}{n}} \right\}$$

Rejection Region: Example: psychiatrist problem

- $\alpha = 0.05$

$$R = \{(x_1, x_2, \dots, x_n) : \bar{x} < 0.1134\}$$

Reject null hypothesis

-

$$R = \{(x_1, x_2, \dots, x_n) : \bar{x} < 0.0984\}$$

Accept null hypothesis

Example: psychiatrist problem: Decision

- p-value= 0.0359 represents the probability to observe a sample mean with a more extreme value than 0.11
- If $\alpha = 0.05$, we reject null hypothesis, coke does have effect on time reaction (coke decreases reaction time)
- If $\alpha = 0.01$, we DO NOT reject null hypothesis, coke does NOT have effect on time reaction

Application to Poisson Distribution

Let $(X_1, \dots, X_n)$ be a random sample of i.i.d. random variables distributed as a Poisson distribution with parameter λ

$$f(x; \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!}$$

We wish to test $H_0 : \lambda = 6$ versus $H_1 : \lambda \neq 6$

We observe a random sample $(X_1, \dots, X_{100})$, such that $\sum_i x_i = 550$

Application to Poisson Distribution

Observe that the joint pdf of $X = (X_1, \dots, X_n)$ is give by

$$f(x_1, \dots, x : n; \lambda) = \prod_i \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!}$$

$$L(\lambda) = \frac{\lambda^{\sum_i x_i} \exp(-n\lambda)}{\prod_i x_i!}$$

$$l(\lambda) = \sum_i x_i \log(\lambda) - n\lambda - \log \left(\prod_i x_i! \right)$$

$$\frac{dl(\lambda)}{\lambda} = \frac{\sum_i x_i}{\lambda} - n$$

$$\frac{d^2 l(\lambda)}{\lambda^2} = -\frac{\sum_i x_i}{\lambda^2}$$

$$\hat{\lambda} = \bar{x} = 5.5$$

Application to Poisson Distribution

$$l(\lambda) = \sum_i x_i \log(\lambda) - n\lambda - \log \left(\prod_i x_i! \right)$$

$$\frac{dl(\lambda)}{\lambda} = \frac{\sum_i x_i}{\lambda} - n$$

$$\frac{d^2 l(\lambda)}{\lambda^2} = -\frac{\sum_i x_i}{\lambda^2}$$

$$I_n(\lambda) = -E \left(\frac{d^2 l(\lambda)}{\lambda^2} \right) = -\frac{n}{\lambda}$$

$$\hat{\lambda} = \bar{x}$$

Poisson- Large sample test

- WALD Test statistics

$$\frac{\bar{x} - \lambda_0}{\sqrt{\frac{\bar{x}}{n}}}$$

approximate distribution $N(0, 1)$



$$\frac{\bar{x} - \lambda_0}{\sqrt{\frac{\bar{x}}{n}}} = \frac{5.5 - 6}{\sqrt{\frac{5.5}{100}}} = -2.13$$

p-value=0.033

Poisson- Large sample test

- Score test statistics

$$\frac{\frac{\sum_i x_i}{\lambda_0} - n}{\sqrt{\frac{n}{\lambda_0}}}$$

approximate distribution $N(0, 1)$

-

$$\frac{\frac{\sum_i x_i}{\lambda_0} - n}{\sqrt{\frac{n}{\lambda_0}}} = -2.04$$

p-value=0.041

Poisson- Large sample test

- Likelihood Ratio Test Statistics

$$\begin{aligned}2 \times \left(l(\hat{\lambda}) - l(\lambda_0) \right) &= 2 \times \left(\sum_i x_i \log \frac{\hat{\lambda}}{\lambda_0} - n(\hat{\lambda} - \lambda_0) \right) \\ \Lambda(x) &= 2 \times \left(\sum_i x_i \log \frac{\bar{x}}{\lambda_0} - n(\bar{x} - \lambda_0) \right) \\ &= 4.29 \\ p - value &= 0.038\end{aligned}$$

approximate distribution χ_1