Probability:
Problem Set 2
Solutions

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1 [The birthday problem]

Solution

The right answer is 23 (b).

It is simpler to calculate $P(A_n^c)$, that is the probability of there not being any two people having the same birthday. Define the events

$P(i) = \{$person i does not share his/her birthday with previously considered people$\}$,

$i = 1, \ldots, n$.

Given person 1,

- $P(2) = \frac{364}{365}$,
- $P(3) = \frac{363}{365}$,

  $\vdots$

- $P(n) = \frac{365-n+1}{365}$.

Since the events are independent,

$$P(A_n^c) = \frac{364}{365}\frac{363}{365}\cdots\frac{365-n+1}{365} = \frac{364!}{365^{n-1}(365-n)!}.$$

It follows that

$$P(A_n) = 1 - P(A_n^c) = 1 - \frac{364!}{365^{n-1}(365-n)!}.$$

Trying different values of n, we get that the smallest n is 23.

2 [The False positive problem]

Solution

The right answer is $< 8\%$ (c).

Suppose that the population consists of 1000 individuals. The number of individuals who really have allergy is

$$\frac{1}{100}1000 = 10.$$

The number of individuals who get a True positive is

$$\frac{79}{100}10 = 7.9 \approx 8.$$

The number of individuals who get a False positive is

$$\frac{10}{100}990 = 99.$$

Therefore out of 1000 people the test says "Yes" to 8+99 = 107 people. So 107 people get a "Yes" but only 8 of those really have the allergy. Therefore the probability that you are really infected is

$$\frac{8}{107} \approx 0.07 = 7\%.$$

3 [**The Monty Hall problem**]

Solution

The right answer is that it is better to switch to (a).

There are three equally possible scenarios, each of them having probability 1/3.

1. The player picks the goat 1. The host picks goat 2. Behind the third door there is the car.

2. The player picks the goat 2. The host picks the goat 1. Behind the third door there is the car.

3. The player picks the car. The host picks a goat. Behind the third door there is the other goat.

In the scenarios 1 and 2, if the player switches then he/she wins the car. In the scenario 3 if the player switches then he/she does not win. Therefore the probability of winning by switching is 2/3, while probability of winning by staying with the initial choice is 1/3.

4 [**The queue problem**]

Solution

The right answer is that it does not matter (c).

Define the events

$$S_i = \{\text{the soldier } i \text{ picks the longer stick}\},$$

$i = 1, \ldots, n$.

- $P(S_1) = \frac{1}{n}$,
- $P(S_2) = P(S_2 \cap S_1) + P(S_2 \cap S_1^c) = P(S_2 \mid S_1^c)P(S_1^c) = \frac{1}{n-1}(1 - P(S_1)) = \frac{1}{n-1}\left(1 - \frac{1}{n}\right) = \frac{1}{n}$,
- $P(S_3) = P(S_3 \cap (S_1 \cup S_2)) + P(S_2 \cap (S_1 \cup S_2)^c) = P(S_2 \mid (S_1 \cup S_2)^c)P(S_1 \cup S_2)^c = \frac{1}{n-2}(1 - P(S_1 \cup S_2)) = \frac{1}{n-2}(1 - P(S_1) - P(S_2) + P(S_1 \cap S_2)) = \frac{1}{n}$,

and so on.