Giulia Pavan, based on Chiara Perricone's notes

Calculus

Problem Set 2
Solutions

$\hookrightarrow$ **Topics**

Limits, polar coordinates, continuity, gradient, first and second partial derivatives, Schwartz's theorem, Hessian matrix, directional derivatives, differentiability, gradient, level curves, directional derivatives.

Exercise 1

Exercise 2 Written examination October, 2012

Exercise 2

Compute the following limits:

-
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

In polar coordinates:

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$

we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} &= \lim_{\rho\to0^+} \frac{\rho\cos\theta\rho^2\sin^2\theta}{\rho^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}} \\ &= \lim_{\rho\to0^+} \rho\underbrace{\cos\theta\sin^2\theta}_{\textit{bounded}} = 0.\end{aligned}$$

Indeed,

$$\left|\rho\cos\theta\sin^2\theta\right| \le \rho.$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3-2xy+y^2}{x^2+y^2}.$$

In polar coordinates

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3-2xy+y^2}{x^2+y^2} &= \lim_{\rho\to0^+} \frac{\rho^3\cos^3\theta-2\rho^2\cos\theta\sin\theta+\rho^2\sin^2\theta}{\rho^2} \\ &= \lim_{\rho\to0^+} \rho\cos^3\theta-2\cos\theta\sin\theta+\sin^2\theta,\end{aligned}$$

which is clearly not indenpendent of θ. This suggests that the limit does not exist. Indeed, let $y = mx$ and compute the limit on the line passing through $(0,0)$:

$$
\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^3 - 2xy + y^2}{x^2+y^2} &= \lim_{x\to 0} \frac{x^3 - 2x(mx) + (mx)^2}{x^2 + (mx)^2} \\
&= \lim_{x\to 0} \frac{x^3 - 2mx^2 + m^2x^2}{x^2(1+m^2)} \\
&= \lim_{x\to 0} \frac{x - 2m + m^2}{1+m^2} \\
&= \frac{m^2 - 2m}{1+m^2}.
\end{aligned}
$$

The limit changes depending on m.

- Verify if the following function is continuous in the point $(0,0)$

$$
f(x,y) = \begin{cases} \frac{(y^2-x^2)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}.
$$

In polar coordinates:

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{\rho\to 0^+} \frac{(\rho^2 \sin^2\theta - \rho^2\cos^2\theta)^2}{\rho^2\sin^2\theta + \rho^2\cos^2\theta} \\
&= \lim_{\rho\to 0^+} \frac{\rho^4(\sin^2\theta - \cos^2\theta)^2}{\rho^2} \\
&= \lim_{\rho\to 0^+} \rho^2(\sin^2\theta - \cos^2\theta)^2 = 0 \text{ (uniformly, w.r.t. } \theta) \\
&\neq f(0,0).
\end{aligned}
$$

The function is not continuous in $(0,0)$.

- Check the continuity and the differentiability at $(0,0)$ of $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$
f(x,y) = \sqrt{x^2+y^2}.
$$

Continuity: the function is continuous because it is the composition of continuous functions.

Differentiability

Consider the function along the x axis, $y = 0$
$f(x, 0) = \sqrt{x^2} = |x|$ the absolute value function is continuous in $(0, 0)$ but not differentiable.

It is the same if you consider the function along the y axis, $x = 0$
$f(0, y) = \sqrt{y^2} = |y|$
Therefore, the function is continuous but not differentiable in the point.

Exercise 3. Compute the gradient of the following functions:
$f(x, y) = x^2 + 2xy - y^2$
$f_x \equiv \frac{\partial f}{\partial x} = 2x + 2y$
$f_y \equiv \frac{\partial f}{\partial y} = 2x - 2y$
Thus, $\nabla f(x, y) = (2x + 2y, 2x - 2y)$.

$f(x, y) = y^2 e^{-x}$
$f_x = \frac{\partial f}{\partial x} = -y^2 e^{-x}$
$f_y = \frac{\partial f}{\partial y} = 2ye^{-x}$
Thus, $\nabla f(x, y) = (-y^2 e^{-x}, 2ye^{-x})$.

Exercise 4
Verify that the first order partial derivative exists in (x_0, y_0), compute it and verify it is continuous in (x_0, y_0)
$f(x, y) = \begin{cases} \frac{x^3 y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$
in $(0, 0)$

$$f_x(0,0) \equiv \frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^3 0 - (\Delta x) 0}{(\Delta x)^2 + 0} + 0}{\Delta x} = 0,$$
$$f_y(0,0) \equiv \frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{0(\Delta y) - 0(\Delta y)}{0 + (\Delta y)^2} + 0}{\Delta y} = 0$$

Therefore, the derivative exists in $(0, 0)$ Compute the first order derivatives. The results are:

$$\frac{\partial f}{\partial x} = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$
$$\frac{\partial f}{\partial y} = \frac{-xy^4 - 4x^3 y^2 - x^5}{(x^2 + y^2)^2}$$

To verify the continuity in $(0,0)$, follow the solutions of exercise 1, substituting x and y with the polar coordinates

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$

Exercise 5 Compute the first and second partial derivatives, write the Hessian matrix and check the validity of Schwartz's theorem for the following functions:

$f(x,y) = \frac{1}{\sqrt{7x+4y-2}}$

$\frac{\partial f}{\partial x}(x,y) = -\frac{7}{2}(7x+4y-2)^{-\frac{3}{2}}$

$\frac{\partial f}{\partial y}(x,y) = -2(7x+4y-2)^{-\frac{3}{2}}$

$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{147}{4}(7x+4y-2)^{-\frac{5}{2}}$

$\frac{\partial^2 f}{\partial y^2}(x,y) = 12(7x+4y-2)^{-\frac{5}{2}}$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 21(7x+4y-2)^{-\frac{5}{2}} = \frac{\partial^2 f}{\partial x \partial y}(x,y)$

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{147}{4}(7x+4y-2)^{-\frac{5}{2}} & 21(7x+4y-2)^{-\frac{5}{2}} \\ 21(7x+4y-2)^{-\frac{5}{2}} & 12(7x+4y-2)^{-\frac{5}{2}} \end{bmatrix}$$

Exercise 6

Compute the directional derivatives of the following function:

$f(x,y) = x^2 + xy - 2$ in $P(1,0)$ along the direction $\vec{v} = (2,1)$

Note that $f \in C^\infty(\mathbb{R}^2;\mathbb{R})$, since a polynomial is everywhere infinite continuously differentiable, in particular f is differentiable in $(1,0)$. Hence, we can compute the directional derivatives as the scalar product between the gradient in $(1,0)$ and the vector $\vec{v}$. Now,

$$\nabla f(x,y) = (2x+y, x) \Rightarrow \nabla f(1,0) = (2,1).$$

Therefore, the directional derivative is given by:

$$\frac{\partial f}{\partial v}(1,0) = \nabla f(1,0) \cdot \vec{v} = (2,1) \cdot (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \sqrt{5}.$$

More generally, you can use the definition

$$f_v(x_0,y_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$