Giulia Pavan

Calculus and Linear Algebra
Practice October, 25th

$\hookrightarrow$ **Topics**
Tangent planes, differentiability, gradient, level curves, directional derivatives. Eigenvalues and eigenvectors. Double integrals.

Exercise 1

Find the tangent plane of the following function:

- $f(x, y) = x^3 - y^3$ in $P(0, 1, -1)$.
- $f(x, y) = x^y + y^x$ in $P(1, 1, 2)$.

Solution
$f(x, y)$ is differentiable in $(x_0, y_0) \Leftrightarrow \exists$ a tangent plane in $(x_0, y_0, f(x_0, y_0)$ to $f(x, y)$ tangent plane equation:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Consider the function $f(x, y) = x^3 - y^3$ in $P(0, 1, -1)$.

$z = f(0,1)+f_x(0,1)(x-0)+f_y(0,1)(y-1) = -1+3(0)^2x-3(1)^2(y-1) = -1-3y+1 = 2-3y$

Consider now the function $f(x, y) = x^y + y^x$ in $P(1, 1, 2)$.

$z = f(1,1)+f_x(1,1)(x-1)+f_y(1,1)(y-1) = 2+1(x-1)+1(y-1) = 2+x-1-y-1 = x+y$

Exercise 2

Check that the
gradient is orthogonal to the correspondent level curve when $f(x, y)$ is:

$$f(x, y) = y - x$$

$L_c = \{(x, y) \in \mathbb{R}^2 \mid y - x = c\}$ This is the set of straight line with slope 1
Gradient

$\bigtriangledown f(x,y) = (-1,1), \forall(x,y)$ Therefore, the gradient is orthogonal to the correspondent level curve.

Exercise 3

Check that the
gradient is orthogonal to the level curve $L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 1\}$ at the points $P(0,1)$ and $P'(0,-1)$ when $f(x,y)$ is:

$$f(x,y) = y^2 - x^2$$

$L_c = \{(x,y) \in \mathbb{R}^2 \mid y^2 - x^2 = c\}$
consider c=1, the hyperbola you get intersect the y axis at P and P'

The gradient is $\bigtriangledown f(x,y) = (-2x, 2y)$

At P it is $\bigtriangledown f(x,y) = (0,2)$

At P' it is $\bigtriangledown f(x,y) = (0,-2)$

Exercise 4

Consider the function $f(x,y) = x^2 + 2xy + 2y^2$ in the neighborhood of the point $(2,1)$. Determine a direction v in which the directional derivative $\frac{\partial f}{\partial v}$ at the point $(2,1)$ is null.

Solution

$$\nabla f(x,y) = (2(x+y), 2(x+2y)) \qquad \Rightarrow \qquad \nabla f(2,1) = (6,8)$$

Therefore for a direction $v = (v_1, v_2)$ one has

$$0 = \frac{\partial f}{\partial v}(2,1) = \langle \nabla f(2,1), v \rangle = \langle (6,8), (v_1, v_2) \rangle = 6v_1 + 8v_2$$

so that $v_2 = -\frac{3}{4}v_1$ and $v_1^2 + v_2^2 = 1$.
Therefore,

$$(v_1, v_2) = \left(\pm\frac{4}{5}, \mp\frac{3}{5}\right)$$

Exercise 5
Compute the eigenvalues of the following matrices:

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}; B = \begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix}; C = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}.$$

Compute the eigenvectors corresponding to the eigenvalues for matrix C.
Solution

We are looking for a λ such that $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

$$(5-\lambda)(2-\lambda) - 4 = 0; \lambda^2 - 7\lambda + 6 = 0;$$

Therefore, $\lambda_1 = 6$ and $\lambda_2 = 1$.

Notice that this matrix is singular, we expect that one λ is equal to zero

$$\det(B - \lambda I) = \begin{vmatrix} 3-\lambda & 6 \\ 9 & 18-\lambda \end{vmatrix} = 0.$$

$$(3-\lambda)(18-\lambda) - 54 = 0; \lambda^2 - 21\lambda = 0;$$

Therefore, $\lambda_1 = 21$ and $\lambda_2 = 0$.

$$\det(C - \lambda I) = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{vmatrix} = 0.$$

$(6-\lambda)(5-\lambda)(7-\lambda)-2(5-\lambda)2-(-2)(7-\lambda)(-2) = 0; \lambda^3+18\lambda^2-99\lambda+162 = 0$

$(\lambda-3)(\lambda-6)(-\lambda+9) = 0$

Therefore, $\lambda_1 = 9$, $\lambda_2 = 6$ and $\lambda_3 = 3$.

An eigenvector of a square matrix A is a non-zero vector v that, when the matrix is multiplied by v, yields a constant multiple of v, the multiplier is the eigenvalue. That is:

$$Cv = \lambda v$$

Let's consider λ_1

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$(C - \lambda I)v = \mathbf{0}$

$$\begin{pmatrix} -3 & -2 & 2 \\ -2 & -4 & 0 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x - 2y + 2z = 0 \\ -2x - 4y = 0 \\ 2x - 2z = 0 \end{cases}$$

The eigenvector corresponding to the eigenvalue $\lambda_1 = 9$ is $(x, -\frac{1}{2}x, x)$.

Exercise 6

Exercise 6

Let $D \subset \mathbb{R}^2$ the bounded region delimited by the curves $y = x^2$, $y = 0$ and $x = 2$.

Calculate in two different ways. $\int\int_D (x^2 + y^2) dx dy$

$$\int_0^4 \left[\int_{\sqrt{y}}^2 (x^2 + y^2) dx \right] dy = \int_0^4 \left[\frac{x^3}{3} + y^2 x \right]_{\sqrt{y}}^2 = \int_0^4 \left[\frac{8}{3} + y^2 2 - \frac{y^{\frac{3}{2}}}{3} - y^2 \sqrt{y} \right]_{\sqrt{y}}^2 dy =$$

$$= \left[\frac{8}{3} y + \frac{y^3}{3} 2 - \frac{1}{3} \frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^4 = \frac{8}{3} 4 + \frac{64}{3} 2 - \frac{1}{3} 2^{\frac{5}{2}} \frac{2}{5} - \frac{2}{7} 2^{\frac{7}{2}} =$$

$$= \frac{32}{3} + \frac{128}{3} - \frac{1}{3} \frac{32}{5} 2 - \frac{128}{7} 2 = \frac{1312}{105}$$

Second way:

$$\int_0^2 \left[\int_0^{x^2} (x^2 + y^2) dy \right] dx = \int_0^2 \left[\frac{y^3}{3} + y x^2 \right]_0^{x^2} = \int_0^2 \left(\frac{x^6}{3} + x^4 \right) dx =$$

$$= \left[\frac{1}{3} \frac{x^7}{7} + \frac{x^5}{5} \right]_0^2 = 32 \left(\frac{4}{21} + \frac{1}{5} \right) = \frac{1312}{105}$$