Surname ........................................ First name ........................................
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**Mathematics – Teacher: Paolo Gibilisco**

**(Simulation 2 of) Written Examination**

**Rules**: you cannot use any textbook, lecture notes, neither any electronic device (very welcome a traditional watch). Write your solutions on the enclosed sheets. You will receive additional sheets for the preliminary drafts. The solutions of the exercises must be detailed. One of the exercises requires the proof of a theorem. For the quizzes you have simply to indicate your choice in the True-False alternative. You have **two hours and half** to finish your work. An **identity card** or a passport is needed to participate.

**Quiz 1.** *Correct answer: points 2. Wrong answer: points -1. No answer: points 0.*
The change of variable formula for polar coordinates is the following

$$\int\int_{\mathbb{R}^2} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^{\pi}\int_0^{+\infty} f(\rho\cos\theta, \rho\sin\theta)\rho^2\,\mathrm{d}\rho\mathrm{d}\theta$$

TRUE FALSE

**Quiz 2.**

$$\mathrm{Im}(e^i) = \sin(1)$$

TRUE FALSE

**Quiz 3.**
If $p(\lambda) = \lambda^4 - 1$ is the characteristic polynomial of $4 \times 4$ matrix $A$ then $A$ is not symmetric.

TRUE FALSE

**Quiz 4.**
If $f$ is differentiable in $(2,3)$ and $\nabla f(2,3) = (1,2)$ then the directional derivate $\frac{\partial f}{\partial v}(2,3)$ assumes its maximum value in for $v = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.

TRUE FALSE

**Quiz 5.**
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function and let its Hessian matrix in the stationary point (0,0) be equal to

$$H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we can say that (0,0) is a local minimum point for $f$.

TRUE FALSE

**Quiz 6.**
$E(Y|X) = E(Y) \Longrightarrow X \perp Y$

TRUE FALSE

**Quiz 7.**

The exponential distributions is a particular case of the Gamma distribution.

TRUE FALSE

**Exercise 1.** *6 points*

Let

$$E = \{(x, y) \in \mathbb{R}^2 | x \in [1, 4] \quad 1 \le y \le e^x\}$$

Calculate

$$\int\int_E \frac{\sqrt{x}}{y} \, \mathrm{d}x\mathrm{d}y$$

**Exercise 2.** *6 points*

Find the spectral decomposition of the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

**Exercise 3.** *6 points*

Maximize the function

$$f(x, y) = xy$$

on the constraint set

$$g(x, y) = x^2 + y^2 \le 1\,.$$

**Exercise 4.** *6 points*

$(X, Y)$ is a random vector with uniform density on $A = \{(x, y) \in R^2 | x^2 + y^2 \le 1\}$.
Let $U := X + Y$.
i) Calculate $F_U(-2)$, $F_U(0)$, $F_U(2)$.
ii) Find the marginal densities $f_X, f_Y$.
iii) Are $X$ and $Y$ independent?

**Exercise 5.** *6 points*
State the Brouwer's fixed point theorem in $\mathbb{R}^n$. Prove it for $n = 1$ and $A = [0, 1]$.