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Mathematics – Teacher: Paolo Gibilisco

(Simulation 3 of) Written Examination

Rules: you cannot use any textbook, lecture notes, neither any electronic device (very welcome a traditional watch). Write your solutions on the enclosed sheets. You will receive additional sheets for the preliminary drafts. The solutions of the exercises must be detailed. One of the exercises requires the proof of a theorem. For the quizzes you have simply to indicate your choice in the True-False alternative. You have **two hours and half** to finish your work. An **identity card** or a passport is needed to participate.

Quiz 1. *Correct answer: points 2. Wrong answer: points -1. No answer: points 0.*
The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x} 1_{[1,e]}(x)$$

is a density.

TRUE FALSE

Quiz 2.

$$\int \int_{\mathbb{R}^2} e^{-(x^2+y^2)} \mathrm{d}x \mathrm{d}y = \pi$$

TRUE FALSE

Quiz 3.
The matrix

$$B = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

is orthogonal.

TRUE FALSE

Quiz 4. Let A be a $(2n+1) \times (2n+1)$ real matrix. Then A must have at least a real eigenvalue.

TRUE FALSE

Quiz 5.
Let $A = \mathbb{R}^2/(0,0)$. A is not closed and not bounded. Let $f : A \to \mathbb{R}$ be a continuous function. The Weierstrass theorem implies that f cannot have a global maximum and minimum.

TRUE FALSE

Quiz 6.

If $X, Y \sim \mathcal{N}(0, \sigma^2)$ then the random variable $\frac{X}{\sqrt{X^2+Y^2}}$ does not depend on σ.

TRUE FALSE

Quiz 7.
If $X \sim$ Poisson(λ) then $P(X \leq 0) > 0$.

TRUE FALSE

Exercise 1. *6 points*

Using polar coordinates calculate

$$\int\int_A 2y \, \mathrm{d}x\mathrm{d}y$$

where

$$A = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0, \, (x-1)^2 + y^2 < 1\}.$$

Exercise 2. *6 points*

Find the spectral decomposition of

$$A = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ \\ \frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

and calculate $\sqrt{A}$.

Exercise 3. *6 points*

Consider the function

$$f(x, y) = \log\left(\frac{x(y^2+1)}{y(x^2+1)}\right).$$

Find: i) the domain; ii) the stationary points; iii) the character of the stationary points (local max, min, saddle).

Exercise 4. *6 points*

Let us consider a Gaussian vector

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 2 \\ a \end{pmatrix} \begin{pmatrix} 3 & b \\ -1 & 1 \end{pmatrix}\right)$$

such that $\mathrm{E}(XY) = 2$.
i) Calculate a e b.
ii) For which values of c and d are the random variables $dX - cY$ and X independents?

Exercise 5. *6 points*

Let A be a $n \times n$ matrix such that $A = A^t$. Let λ, μ be eigenvalues of A such that $\lambda \neq \mu$. Prove that V_λ and V_μ are orthogonal.