Microeconomics for Business
Practice Session 2 - Solutions

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Exercise 1. *In a market there are two firms, firm 1 and firm 2. Each firm produces a good. The two goods are imperfect substitutes. Indeed, the demand function for each good is:*

$$\begin{aligned} Q_1(P_1; P_2) &= 10 - \alpha P_1 + P_2 \\ Q_2(P_1; P_2) &= 10 + P_1 - \alpha P_2 \end{aligned}$$

where P_i is the price of the good produced by firm i. Higher degree of substitutability corresponds to higher α. Assume that $\alpha > 1$, and that both firms have a total cost function of $C_i(q_i) = 0$, $\forall q_i \geq 0$.
The firms maximize their profits and they simultaneously set the prices of the goods. Determine the Nash equilibrium and the level of profits for both firms. How do changes in α affect profits in equilibrium?

Solution 1. The set of players is: {firm 1, firm 2}. The set of pure strategies for each firm is $[0, \infty)$. The payoff function for each firm, for a given pair of prices, is:

$$\Pi_i(P_i, P_j) = P_i(10 - \alpha P_i + P_j)$$

with $i = 1, 2$ and $j \neq i$.
Let's characterize the best response function of each firm for a given price of the other firm. Writing the profit maximizing problem of the firm i,

$$\max_{P_i} \Pi_i = \max_{P_i} P_i(10 - \alpha P_i + P_j) = \max_{P_i}(10 + P_j)P_i - \alpha P_i^2, \tag{1}$$

computing the first order conditions of the profit maximization problem of every firm i,

$$\frac{\partial \Pi_i}{\partial P_i} = 10 + P_j - 2\alpha P_i, \tag{2}$$

and setting it equal to zero, then we get

$$\begin{aligned} P_1(P_2) &= \frac{10 + P_2}{2\alpha} \\ P_2(P_1) &= \frac{10 + P_1}{2\alpha}. \end{aligned}$$

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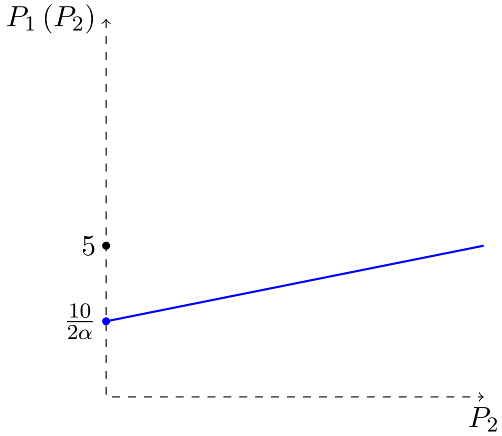
Figure 1: Exercise 5 - Best response of firm 1



The equilibrium quantities can be characterized solving the following system:

$$\begin{cases} P_1 & = & \dfrac{10+P_2}{2\alpha} \\ P_2 & = & \dfrac{10+P_1}{2\alpha} \end{cases}$$

By substitution of the second expression in the first we obtain:

$$P_1 = \frac{10 + \frac{10+P_1}{2\alpha}}{2\alpha} = 10\frac{2\alpha+1}{4\alpha^2} + \frac{P_1}{4\alpha^2} \Leftrightarrow$$
$$\left(1 - \frac{1}{4\alpha^2}\right) P_1 = 10\frac{2\alpha+1}{4\alpha^2} \Leftrightarrow \frac{4\alpha^2-1}{4\alpha^2} P_1 = 10\frac{2\alpha+1}{4\alpha^2} \Leftrightarrow$$
$$P_1 = 10\frac{2\alpha+1}{4\alpha^2-1} = 10\frac{2\alpha+1}{(2\alpha-1)(2\alpha+1)} = \frac{10}{2\alpha-1}$$

and, of course,

$$P_2 = \frac{10+P_1}{2\alpha} = \frac{10}{2\alpha-1}.$$

Since the problem is symmetric, we can predict $P_1^* = P_2^*$ and substitute the equilibrium in one of the best replies, for example the first one:

$$P_1^* = \frac{10+P_1^*}{2\alpha} \Leftrightarrow P_1^* = \frac{10}{2\alpha-1}.$$

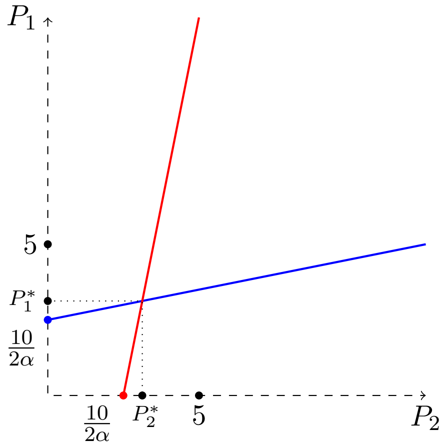
The check with the other best reply restitutes indeed the same price:

$$P_2^* = \frac{10+P_1^*}{2\alpha} = \frac{10}{2\alpha-1}.$$

Thus the unique Nash equilibrium of our game is the pair (P_1^*, P_2^*) with:

$$P_1^* = P_2^* = \frac{10}{2\alpha-1}.$$

Figure 2: Exercise 5 - Best responses of both firms



The profits corresponding to the Nash equilibrium are:

$$\begin{aligned}\Pi_1(P_1^*, P_2^*) &= P_1 Q_1(P_1; P_2)\\ &= \frac{10}{2\alpha - 1}\left(10 - \alpha\frac{10}{2\alpha-1} + \frac{10}{2\alpha-1}\right) = \frac{100\alpha}{(2\alpha-1)^2}\\ \Pi_2(P_1^*, P_2^*) &= P_2 Q_2(P_1; P_2) = \frac{100\alpha}{(2\alpha-1)^2}\end{aligned}$$

To evaluate how changes in α affect profits in equilibrium is sufficient to derive the equilibrium profits with respect to α:

$$\frac{\partial \Pi_i}{\partial \alpha} = 100\frac{1 * (2\alpha-1)^2 - \alpha * 4(2\alpha-1)}{(2\alpha-1)^4} = -100\frac{2\alpha+1}{(2\alpha-1)^3} < 0$$

for $i = 1, 2$.
Hence profits in equilibrium decrease when α increases, i.e. when the degree of substitutability between products increase. Alternatively, without taking derivatives, notice that in the expression for equilibrium profits, $\Pi_i = 100\frac{\alpha}{(2\alpha-1)^2}$, the numerator grows linearly with α whereas the denominator grows quadratically with α. Loosely speaking, as α increases the denominator grows more than the numerator. Hence profits go down as α gets larger.

Exercise 2. *In the following normal-form game, there are three countries: France, Germany and Italy. Let France choose the rows, Germany choose the columns and Italy choose the matrices. Every country simply chooses her economic policy. Define the strategy space for each player.*

	Development	*Austerity*
Development	$5, 5, 5$	$3, 6, 3$
Austerity	$6, 3, 3$	$4, 4, -1$

Development for Italy

What strategies survive iterated elimination of strictly dominated strategies? What are the pure-strategy Nash equilibria? Comment on the results.

	Development	*Austerity*
Development	$3, 3, 6$	$-1, 4, 4$
Austerity	$4, -1, 4$	$0, 0, 0$

Austerity for Italy

Solution 2. The strategy set for every Country is $\{\text{Development}, \text{Austerity}\}$. It is immediate to see that, for every Country, **'Development'** is strictly dominated by **'Austerity'**. For example, the strategy *Development* for France restitutes $u_F(\mathit{Development}, \cdot, \cdot) = (5, 3, 3, -1)$, where the first two elements represent the payoffs when Italy keeps fixed *Development* and Germany chooses *Development* and *Austerity* respectively, whereas the last two elements are the payoffs when Italy chooses *Austerity*. The same convention applied to France chosing *Austerity* brings to $u_F(\mathit{Austerity}, \cdot, \cdot) = (6, 4, 4, 0)$. Therefore, the only Nash Equilibrium of the game is the strategy profile in which every Country chooses **'Austerity'**.

Exercise 3. *Assume there are N firms in the Cournot model we discussed in class. Let q_i be the quantity produced by firm i, and let $Q = q_1 + ... + q_N$ be the aggregate quantity offered on the market. Let P be the market price for the good and let the inverse demand function be given by $P(Q) = a - Q$ if $Q < a$, 0 otherwise. Assume that the total cost of firm i to produce the quantity q_i is given by $C(q_i) = cq_i$ with $0 < c < a$, that is the marginal cost is constant and identical for all a firms. Following Cournot assume each firm chooses simultaneously her quantity q_i. Characterize the Nash equilibrium of this game. What happens when N tends to infinity?*

Solution 3. To simplify notation, for every firm i, let $q_{-i} := \sum_{j \neq i} q_j$, that is the sum of the quantities produced by the competitors. Every firm i chooses q_i so as to solve

$$\max_{q_i} \Pi_i = [P(Q) - c] q_i = (a - q_i - q_{-i} - c)\, q_i = (a - c - q_{-i})\, q_i - q_i^2.$$

The solution to this maximization problem is the best response

$$q_i\,(q_{-i}) = \frac{a - c - q_{-i}}{2}.$$

The Nash Equilibrium of this game is the strategy profile $(q_1^*, q_2^*, \dots, q_n^*)$ where, for every i, $q_i^* = q_i\left(q_{-i}^*\right)$. Due to the symmetry of the game[1], we have that firms will produce the same quantity q^* at equilibrium, that is, $q_1^* = q_2^* = \cdots = q_n^* = q^*$. This implies that $q_{-i}^* = (n-1)\, q^*$ and that, for every i,

$$q_i^* = q_i\left(q_{-i}^*\right) = \frac{a - c - (n-1)\, q^*}{2}.$$

Since $q_i^* = q^*$, the latter equation is equivalent to

$$q^* = \frac{a - c - (n-1)\, q^*}{2},$$

which solves for $q^* = \frac{a-c}{n+1}$.
Therefore, for every firm, $q_i^* = \frac{a-c}{n+1}$. Clearly, as n goes to $+\infty$, q_i^* goes to zero.
The aggregate output at equilibrium is $Q^* = nq^* = \frac{n}{n+1}(a - c)$.
The equilibrium price is $P(Q^*) = a - Q^* = \frac{a}{n+1} + \frac{n}{n+1} c$. As n goes to $+\infty$, the market price converges to the marginal cost c.
Individual profits at equilibrium are $\Pi_i^* = (a - Q^* - c)\, q^* = \frac{(a-c)^2}{(n+1)^2}$. Clearly, ss n goes to $+\infty$, profits go to zero.

[1] Firms have the same cost function.

Exercise 4. *Consider the following game with two players, Sonia and Chris are trying to decide on an evening's entertainment. They must choose to attend either the opera or the final match of the regional basketball championship. Both players would rather spend the evening together than apart. However, Chris would rather be together at the Opera, and Sonia would rather be together at the basketball final. Let the payoffs of the players be given by the following matrix, in which the first element represents Chris' payoff, and the second Sonia's payoff.*

	Opera	*Basketball*
Opera	$2, 1$	$0, 0$
Basketball	$0, 0$	$1, 2$

- *Define the strategy space for each player.*
- *Identify the strictly dominated strategies for every player.*
- *Describe the best replies for every player in the game. What can you conclude?*
- *Characterize <u>all</u> the Nash equilibria of such game.*

Solution 4. Chris is player 1 on the rows and Sonia is player 2 on the columns. Denote S_i the strategy space of player i, with $i = 1, 2$. $S_i = \{O, B\}$ for $i = 1, 2$, where $O = Opera$ and $B = Basketball$.

	O	B
O	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

There are no strictly dominated strategies for any player. Indeed, for Chris $u_1(O, \cdot) = (2, 0)$ and $u_1(B, \cdot) = (0, 1)$ involve no relation of dominace, as well for Sonia where $u_2(\cdot, O) = (1, 0)$ and $u_2(\cdot, B) = (0, 2)$.

The best replies *to pure strategies* can be easily characterized. For Chris O is a best reply to O and B is a best reply to B. For Sonia O is a best reply to O and B is a best reply to B. Since there is a double correspondence of best replies, we can infer the existence of two Nash equilibria in pure strategies, namely $NE = \{(O, O), (B, B)\}$.

But the game cpould have also other Nash equilibria in mixed strategies. Let's call r and q the probability associated to playing O for player 1 (Chris) and 2 (Sonia) respectively. Remember that the mixed strategy $(1, 0)$ identifies the pure strategy O and $(0, 1)$ the pure strategy B. A Nash equilibrium is a strategy profile such that

$$u_1(r^*, q^*) \geq u_1(r, q^*), \ \forall r \in [0, 1] \text{ and } u_2(r^*, q^*) \geq u_2(r^*, q), \ \forall q \in [0, 1] \, .$$

Now the best response of player 1 depends on q, the belief he has about player 2's playing. For a given q, player 1 weakly prefers to play O over B when

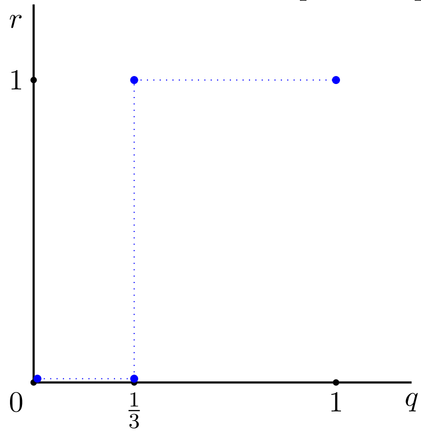
$$\begin{gathered} u_1(O, q) \geq u_1(B, q) \\ q * 2 + (1 - q) * 0 \geq q * 0 + (1 - q) * 1 \\ 2q \geq 1 - q \\ q \geq \frac{1}{3}. \end{gathered}$$

The last expression means that player 1 will choose O if he believes $q > \frac{1}{3}$, B if $q < \frac{1}{3}$, and is indifferent between the two if $q = \frac{1}{3}$. The best response, call it R_i, reads

$$R_1(q) = \begin{cases} O & (r = 1) & \text{if } q > \frac{1}{3} \\ \{O, B\} & (0 \leq r \leq 1) & \text{if } q = \frac{1}{3} \\ B & (r = 0) & \text{if } q < \frac{1}{3} \end{cases}$$

which means that player 1 is willing to randomize between O and B if player 2 uses the mixed strategy $(q = \frac{1}{3}, 1 - q = \frac{2}{3})$. Note that when $q = \frac{1}{3}$, O and B give to player 1 exactly the same expected payoffs, in this sense player 1 is indifferent among all values of r. We can graph the best reply through a cartesian diagram with r (the dependent variable) on the y-axis and q on the x-axis.

Figure 3: Exercise 4 - Best response of player 1



The best response of player 2 is built with the same procedure, that is

$$\begin{gathered} u_2(r, O) \geq u_2(r, B) \\ r * 1 + (1 - r) * 0 \geq r * 0 + (1 - r) * 2 \\ r \geq 2 - 2r \\ r \geq \frac{2}{3}, \end{gathered}$$

and leads to

$$R_2(r) = \begin{cases} O & (q = 1) & \text{if } r > \frac{2}{3} \\ \{O, B\} & (0 \leq q \leq 1) & \text{if } r = \frac{2}{3} \\ B & (q = 0) & \text{if } r < \frac{2}{3} \end{cases},$$

with the same interpretation of player 1's best reply. The graph follows.
Combining the best responses in a single graph, being careful in translating R_2, we can see three strategy profiles (r^*, q^*) where the best replies intersect: $r^* = 1$ and $q^* = 1$, $r^* = 0$ and $q^* = 0$, and $r^* = \frac{2}{3}$ and $q^* = \frac{1}{3}$). All are Nash equilibria (NE) of the game. The first two are the pure strategy NE found before, the third one is a NE in mixed strategies. The same conclusion can be drawn from the analitical best replies. Let's focus on the Mixed NE: each mixed strategy is a best reply to the opponent's mixed strategy. In the end, the set of NE in terms of probability is $NE = \{(1, 1), (0, 0), (\frac{2}{3}, \frac{1}{3})\}$. Alternatively, you can also write $NE = \{(O, O), (B, B), (\frac{2}{3}O + \frac{1}{3}B, \frac{1}{3}O + \frac{2}{3}B)\}$.

Figure 4: Exercise 4 - Best response of player 2

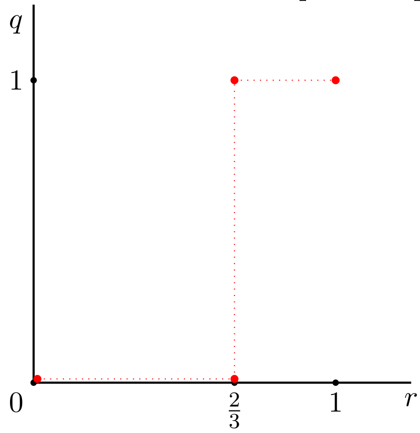


Figure 5: Exercise 4 - Intersection of best responses