Microeconomics for Business
Practice Session 3 - Solutions

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Exercise 1. Show that there are no mixed-strategy Nash equilibria in the Prisoners' Dilemma

	Mum	Fink
Mum	$-1, -1$	$-9, 0$
Fink	$0, -9$	$-6, -6$

and in

	L	M	R
U	1,0	1,2	$0, 1$
D	$0, 3$	$0, 1$	$2, 0$

Solution 1. Let's start with the Prisoners' Dilemma. The set of pure strategies for player $i = 1, 2$ is $S_i = \{M, F\}$. In order to find all the Nash equilibria of the game, call r the probability that player 1 selects M and q that of player 2. The best reply of player 1 can be built by asking him according to which belief about q he plays M, that is when

$$\begin{aligned} u_1(M, q) &\geq u_1(F, q) \\ -1 * q + (-9) * (1 - q) &\geq 0 * q + (-6) * (1 - q) \\ -9 + 8q &\geq -6 + 6q \\ q &\geq \frac{3}{2}. \end{aligned} \tag{1}$$

But q is a probability and cannot be greater than one. So, for any $q \in [0, 1]$ the best reply of player 1 is F or, equivalently, $r = 0$.
The same conclusion can be drawn from figure 1. The utilities associated with a pure strategy, functions of the belief as in eq.1, have been depicted. Playing F delivers a higher utility than M for each belief q. Figure 1 shows the notion of stricly dominated strategy. In this case the efficient frontier (the best reply) corresponds to strategy F.
Player 2 is symmetric to player 1. The best replies can be written as

$$\begin{aligned} B_1(q) = r(q) = 0 \quad \forall q \\ B_2(r) = q(r) = 0 \quad \forall r, \end{aligned}$$

and depicted in figure 2.

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Figure 1: Exercise 1 - Prisoners' Dilemma - Utilities of player 1 associated with his belief about player 2's strategy.

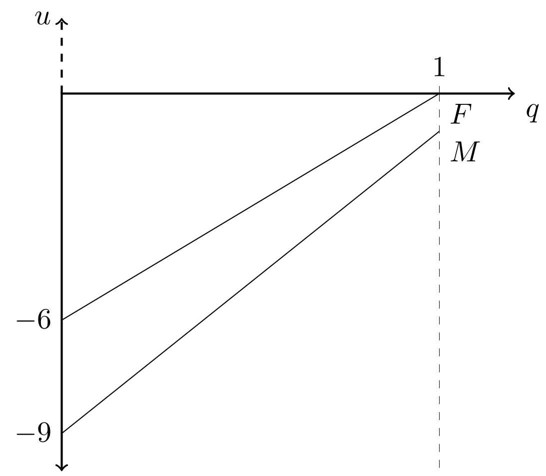
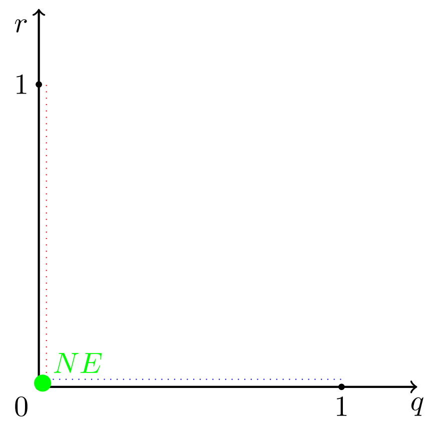


Figure 2: Exercise 1 - Intersection of best responses, player 1's in blue



Hence the mixed strategy profile (r, q) that corresponds to a Nash equilibrium is $(0, 0)$, that is the pure strategy Nash equilibrium (F, F).

The second game is represented in normal form with the following bi-matrix.

	L	M	R
U	$1, 0$	$1, 2$	$0, 1$
D	$0, 3$	$0, 1$	$2, 0$

The mixed strategy of player 1 can be identified with $(r, 1-r)$ while that of player 2 with $q = (q_1, q_2, q_3)$ or also $(q_1, q_2, 1 - q_1 - q_2)$ since $q_3 = 1 - q_1 - q_2$. Following the standard procedure, player 1 weakly prefers the strategy U when

$$\begin{gathered} u(U,q) \geq u(D,q) \\ 1*q_1 + 1*q_2 + 0*(1-q_1-q_2) \geq 0*q_1 + 0*q_2 + 2*(1-q_1-q_2) \\ q_1+q_2 \geq \frac{2}{3}, \end{gathered}$$

or, equivalently, $q_3 \leq \frac{1}{3}$. The best reply is

$$B_1(q) = \begin{cases} r=1 & (U) & \text{if } q_1+q_2 > \frac{2}{3} \\ 0 \leq r \leq 1 & (U,D) & \text{if } q_1+q_2 = \frac{2}{3} \\ r=0 & (D) & \text{if } q_1+q_2 < \frac{2}{3} \end{cases}.$$

Now let's look at player 2. Clearly, it is possible to immediately recognize that strategy R is strictly dominated by M. Therefore it is common knoledge that player 2 will attach a null probability to playing R, i.e. $q_3 = 0$. This means that strategy R does not enter in the support (a narrower set of strategies) of the player 2's best reply and, thus, of any Nash equilibrium.
Since $q_3 = 0$, then $q_1 + q_2 = 1 > \frac{2}{3}$. In this case the best reply of player 1 suggests $r = 1$ for any player 2's mixed strategy of the type $(q_1, q_2, q_3) = (q, 1-q, 0)$. Then $r = 1$ becomes common knoledge too, and the utility of player 2, *given* $r = 1$, is

$$u_2(1,q) = 0*q + 2*(1-q) = 2(1-q).$$

When does player 2 maximize his expected utility? When he chooses $q = 0$, i.e. he strictly prefers M instead of L or any randomization between the two strategies. In the end, the only Nash equilibrium is the profile $((r, 1-r), (q_1, q_2, q_3)) = ((1, 0), (0, 1, 0))$, equivalent to (L, M).

In this case we have immediately recognized the relation of stricly dominance between two pure strategies. Some times a pure strategy is stricly dominated by a mixed strategy and this may not be verifiable in the bi-matrix. When possible, a good practice is to depict, for each player separately, the expected utilities of the pure strategies as function of the own belief about other player's mixed strategy (e.g. figure 1). Then it is possible to recognize the dominance relations more easily. In particular, when a pure strategy is neven part of the efficient frontier (i.e. the best reply), then it must be stricly dominated either by a pure strategy or a mixed strategy. For example, substitute the payoff $u_2(R) = (1, 0)$ with $(1.2, 1.2)$ and show that R is dominated by some combination of L and M. If, instead, the R's payoff was $(1.7, 1.7)$ (put it on the graph) the efficient frontier would be made of all the three strategies and the best reply would involve also $q_3 > 0$ for some values of r.

In general, without relying on intuitions or charts, the procedure to write the best response of player 2 is the following. Write the expected utilities:

$$\begin{aligned} u(r,L) &= 0*r + 3*(1-r) = 3-3r \\ u(r,M) &= 2*r + 1*(1-r) = r+1 \\ u(r,R) &= 1*r + 0*(1-r) = r; \end{aligned}$$

then procede with the inequality two-by-two:

$$\begin{aligned} u(r,L) \geq u(r,M) &\Leftrightarrow r \leq \frac{1}{2} \\ u(r,M) \geq u(r,R) &\Leftrightarrow 1 \geq 0 \text{ (always)} \\ u(r,L) \geq u(r,R) &\Leftrightarrow r \leq \frac{3}{4}; \end{aligned}$$

the secon inequality suggests that R is not part of the best response, which reads

$$B_2(r) = \begin{cases} q_1 = 1 & (L) & \text{if } r < \frac{1}{2} \\ q_1 + q_2 = 1 & (L, M) & \text{if } r = \frac{1}{2} \\ q_2 = 1 & (M) & \text{if } r > \frac{1}{2} \end{cases} .$$

Now we can read the players' best replies together: start from an arbitrary condition for $B_2(r)$, for example $q_1 = 1$; this condition brings to read $B_1(q)$ where $q_1 + q_2 = 1 > \frac{2}{3}$, which corresponds to the reply $r = 1$; how player 2 respond to $r = 1 > \frac{1}{2}$? with $q_2 = 1$; since this is different from the assumed starting point ($q_1 = 1$), then there is no correspondence and no Nash equilibria. Now let's start with $q_2 = 1$: in this case the best reply of player 1 is always $r = 1$ and the best reply of player 2 is $q_2 = 1$, which identifies a correspondence of best replies. Starting from $B_2\,(r = 1/2)$ does not restitutes a correspondece. Using the profile $((r, 1-r), (q_1, q_2, q_3))$, we have $NE = ((1, 0), (0, 1, 0))$.

Exercise 2. Consider the following finite version of the Cournot duopoly model in an environment with inverse demand $P(Q) = a - Q$ and cost function cq_i for $i = 1, 2$.
Suppose each firm must choose either half the monopoly quantity, $\frac{q_M}{2} = \frac{a-c}{4}$, or the Cournot equilibrium quantity, $q_c = \frac{a-c}{3}$. No other quantities are feasible. Show that this two-action game is equivalent to the Prisoners' Dilemma: each firm has a strictly dominated strategy, and both are worse off in equilibrium than they would be if they cooperated.

Solution 2. For each firm $i \in \{1, 2\}$, the strategy space is $S_i = \left\{\frac{q_M}{2}, q_c\right\}$. From now on, to simplify notation we use $q_m := \frac{q_M}{2}$ and $\Delta := (a - c)$.
The normal-form representation of this game is provided in the following payoff matrix.

	q_m	q_c
q_m	$\pi_1(q_m, q_m), \pi_2(q_m, q_m)$	$\pi_1(q_m, q_c), \pi_2(q_m, q_c)$
q_c	$\pi_1(q_c, q_m), \pi_2(q_c, q_m)$	$\pi_1(q_c, q_c), \pi_2(q_c, q_c)$

Notice that, for each firm $i \in \{1, 2\}$, payoff $\pi_i(q_1, q_2)$ is the payoff (profit) to firm i when firm 1 is producing q_1 and firm 2 is producing q_2.
Due to the symmetry of the game[1], the following relations hold:

$$\pi_1(q_c, q_c) = \pi_2(q_c, q_c)$$

$$\pi_1(q_m, q_m) = \pi_2(q_m, q_m)$$

$$\pi_1(q_m, q_c) = \pi_2(q_c, q_m)$$

$$\pi_1(q_c, q_m) = \pi_2(q_m, q_c)\,.$$

In addition, recalling that $q_c = \frac{\Delta}{3}$, $q_m = \frac{\Delta}{4}$, and $\pi_i = P(Q)q_i - cq_i = (a - q_i - q_j - c)q_i$ we have

$$\pi_1(q_c, q_c) = (\Delta - 2q_c)\,q_c = \frac{\Delta^2}{9}$$

$$\pi_1(q_m, q_m) = (\Delta - 2q_m)\,q_m = \frac{\Delta^2}{8}$$

$$\pi_1(q_m, q_c) = (\Delta - q_m - q_c)\,q_m = \frac{5}{48}\Delta^2$$

[1] Both firms have the same cost function.

$$\pi_1(q_c, q_m) = (\Delta - q_c - q_m)\, q_c = \frac{5}{36}\Delta^2.$$

Using these results, the payoff matrix of the game is the following.

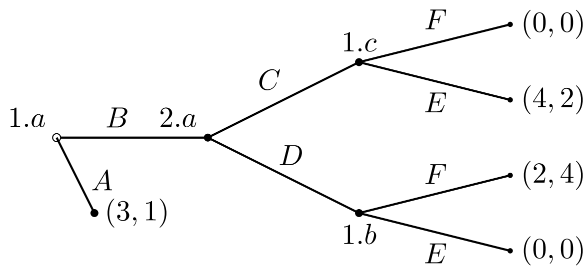
	q_m	q_c
q_m	$\frac{\Delta^2}{8}, \frac{\Delta^2}{8}$	$\frac{5\Delta^2}{48}, \frac{5\Delta^2}{36}$
q_c	$\frac{5\Delta^2}{36}, \frac{5\Delta^2}{48}$	$\frac{\Delta^2}{9}, \frac{\Delta^2}{9}$

Notice that

$$\frac{5}{36} > \frac{1}{8} > \frac{1}{9} > \frac{5}{48}.$$

This implies that this game is equivalent to the Prisoners' Dilemma. In particular, (q_c, q_c) is the only Nash Equilibrium, q_m is strictly dominated by q_c for each firm, and the Nash Equilibrium outcome $\left(\frac{\Delta^2}{9}, \frac{\Delta^2}{9}\right)$ is Pareto dominated by $\left(\frac{\Delta^2}{8}, \frac{\Delta^2}{8}\right)$, which would be the profit if firms cooperated.

Exercise 3. Consider the following extensive form game.



Characterize:

a. the pure-strategy subgame perfect Nash equilibria of the game;

b. write down the normal form representation of the same game, identifying the strategies of every player;

c. compute the Nash equilibria of the game, and compare them with the results in *a*.

Solution 3. a. The strategy set of player 1 is

$$S_1 = \{AEE, AEF, AFE, AFF, BEE, BEF, BFE, BFF\},$$

where the first letter is the action taken at $1.a$, the second is the action at $1.b$, and the third is the action at $1.c$. The strategy set for player 2 is

$$S_2 = \{C, D\}$$

.

To find Subgame Perfect Equilibria we can use backward induction. At node $1.c$, player 1 chooses

E; at node 1.b, player 1 chooses F; at node 2.a, player 2 anticipates players 1's reaction at 1.b and 1.c, hence he selects D. At the initial node 1.a, player 1 anticipates player 2's reaction at 2.a and so he chooses A. Therefore, the unique Subgame Perfect Equilibrium of this game is the strategy profile (AFE, D).

b. The normal-form representation of the game is provided in the following payoff matrix.

	C	D
AEE	$3,1$	$3,1$
AEF	$3,1$	$3,1$
AFE	$3,1$	$3,1$
AFF	$3,1$	$3,1$
BEE	$4,2$	$0,0$
BEF	$0,0$	$0,0$
BFE	$4,2$	$2,4$
BFF	$0,0$	$2,4$

c. The set of Nash Equilibria of the game is

$$NE = \{(AEE, D), (AEF, D), (AFE, D), (AFF, D), (BEE, C)\}.$$

Thus there are 4 Nash Equilibria that are not Subgame Perfect.

Exercise 4. Three oligopolists operate in a market with inverse demand given by $P(Q) = a - Q$, where $Q = q_1 + q_2 + q_3$ and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, c, and no fixed cost. The firms choose their quantities as follows: (1) firm 1 chooses $q_1 \geq 0$; (2) firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3, respectively. What is the subgame-perfect Nash equilibrium?

Solution 4. To find the Subgame Perfect Equilibria of the game we can start by analyzing the subgame between firms 2 and 3 and then we can work backward.
Given q_1, firm 2 chooses q_2 so as to solve

$$\max_{q_2} \quad \Pi_2 = (a - q_1 - q_2 - q_3 - c)\, q_2 = (a - q_1 - q_3 - c)\, q_2 - q_2^2.$$

The solution to this problem is the best response

$$q_2(q_3|q_1) = \frac{a - q_1 - q_3 - c}{2}.$$

As for firm 3, given q_1, it chooses q_3 according to

$$\max_{q_3} \quad \Pi_3 = (a - q_1 - q_2 - q_3 - c)\, q_3 = (a - q_1 - q_2 - c)\, q_3 - q_3^2,$$

which solves for

$$q_3(q_2|q_1) = \frac{a - q_1 - q_2 - c}{2}.$$

The Nash Equilibrium of the subgame between firms 2 and 3 is identified by output levels (q_2^*, q_3^*) such that

$$q_2^* = q_2(q_3^*|q_1) \tag{2}$$

and

$$q_3^* = q_3\left(q_2^*|q_1\right). \tag{3}$$

Both conditions (2) and (3) are satisfied when

$$q_2^* = q_3^* = \frac{a - q_1 - c}{3}.$$

Firm 1 anticipates that firms 2 and 3 will choose q_2^* and q_3^* at equilibrium and it solves

$$\max_{q_1} \quad \Pi_1 = \left(a - q_1 - q_2^* - q_3^* - c\right) q_1 = \left(\frac{a - q_1 - c}{3}\right) q_1 = \frac{1}{3}\left[\left(a - c\right) q_1 - q_1^2\right],$$

which solves for $q_1 = \frac{a-c}{2}$.
Therefore, the Subgame Perfect Equilibrium of the game is the strategy profile $\left(\frac{a-c}{2}, \frac{a-q_1-c}{3}, \frac{a-q_1-c}{3}\right)$.
The corresponding outcome/payoff is $\left(\frac{a-c}{2}, \frac{a-c}{6}, \frac{a-c}{6}\right)$.