

A Proof Without Words and a Maximum Without Calculus for Economists

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July 28, 2014

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Abstract

We define a standard optimization problem with quadratic objective function and provide a rigorous visual proof for its solution without using calculus. We then show that such standard problem is a building block for several economic models related to microeconomics, game theory and pricing strategies.

Keywords: Proof without words, Maximum without calculus, Economic applications.

JEL Classification: A22, C61, C65.

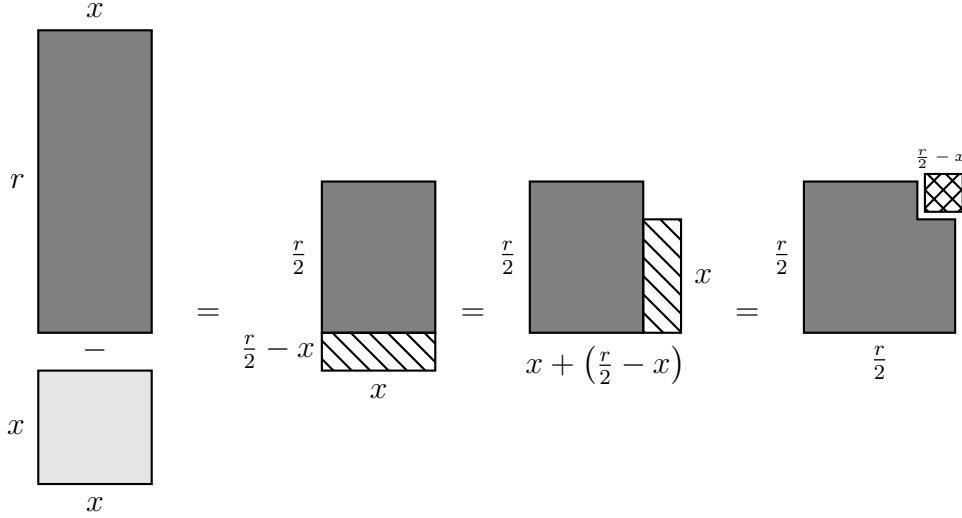
1 Introduction

This paper proposes a unified framework to analyze several topics commonly taught in undergraduate courses in microeconomics, game theory and industrial organization. The key feature of our work is the fact that all the applications we consider are analyzed without relying on derivatives, hence the methodology we propose is very well suited for undergraduate students at the beginning of a major in Economics, when they are still building their mathematical skills, as well as for students with a major in Law, Political Science or Business who are only exposed to basic economic principles.

We start with a simple optimization problem with a quadratic objective function, which is common to all the applications mentioned above. We then show how to solve this problem without resorting to calculus. The solution method is based on a rigorous though very intuitive geometric proof, which only requires to be able to compute and visualize the area of a rectangle. We finally apply such methodology to the several economic examples. In particular, we consider individual decision problems, such as utility maximization by a consumer, profit maximization by a perfectly competitive firm and tax revenue maximization by public authorities. We then discuss the classical oligopoly models á la Cournot, Bertrand and Stackelberg and we finally provide a complete analysis of the pricing strategies of a monopolist, in economies with symmetric and asymmetric information.

2 A Proof Without Words

$$rx - x^2 = \left(\frac{r}{2}\right)^2 - \left(\frac{r}{2} - x\right)^2 \quad (1)$$



(Inspired by Nelsen 1993)

3 A maximum without calculus

The baseline problem we consider is

$$\max_x ax - bx^2 \quad (2)$$

with $a, b > 0$. In what follows, (2) will be referred to as the *standard problem*.

Since $ax - bx^2 = b(rx - x^2)$, with $r = a/b$, and since $b > 0$, the standard problem is equivalent to

$$\max_x rx - x^2$$

As showed in the previous section, the maximum in the above problem is attained when $(r/2 - x)^2 = 0$, hence when $x = r/2$.¹ The solution to the standard problem is therefore $x^* = a/2b$ and its maximum value is $a^2/4b$.

¹See Niven 1981.

4 Applications to economic problems

As mentioned in the introduction, the standard problem often appears in economic courses. To illustrate its applications, we propose several examples related to individual decision making, strategic interactions and pricing strategies.² We describe in detail how to represent each example as a problem in the standard form, so that its unifying role will be clear.

4.1 Individual choice problems

Within this section, three examples are related to consumer's theory: in the first one, we analyze the optimal choice of a consumer with Cobb-Douglas utility function, given her budget constraint. In the second one, we analyze a quantity-tariff problem for a consumer with quasi-linear utility function. In the third one, we describe a portfolio choice problem for a consumer with mean-variance preferences. We then present an example of a firm who maximizes profits in a competitive market for the good she produces and finally discuss a problem of tax revenue maximization in partial equilibrium.

Example 1 (An allocation problem). A consumer wishes to allocate an amount of money m between two commodities, x and y , whose unit prices are p_x and p_y . The consumer values each bundle of goods via the utility function $u(x, y) = xy$. Hence, her problem is to find the bundle with the highest utility among those which are feasible given the monetary endowment. The optimal bundle is, therefore, the solution to the problem

$$\max_{x,y} xy \quad \text{subject to} \quad m = p_x x + p_y y \quad \text{and} \quad x, y \geq 0$$

Observe that $m = p_x x + p_y y$ and $y \geq 0$ imply $0 \leq x \leq m/p_x$. From the first constraint, $y = m/p_y - (p_x/p_y)x$ and, therefore, $xy = (m/p_y)x - (p_x/p_y)x^2$. The above problem is thus equivalent to

$$\max_x \left(\frac{m}{p_y} \right) x - \left(\frac{p_x}{p_y} \right) x^2 \quad \text{subject to} \quad 0 \leq x \leq \frac{m}{p_x} \quad (3)$$

²See, for example, Besanko and Braeutigam 2014 or Varian 2010.

Within the interval of feasible quantity of good x , (3) is a standard problem with $a = m/p_y$ and $b = p_x/p_y$. Since the standard solution $x = a/2b = m/2p_x$ is feasible, we conclude that the optimal bundle is $x^* = m/2p_x$ and $y^* = m/p_y - (p_x/p_y)x^* = m/2p_y$.

Example 2 (A purchase problem). A consumer wishes to purchase some q units of a consumption good in exchange for a total payment of t . Her preferences are represented by the utility function

$$u(q, t) = \theta v(q) - t, \quad \text{with} \quad v(q) = \left(q - \frac{q^2}{2} \right) \quad (4)$$

Let $\theta > 0$ be the individual taste parameter, which captures the intensity of preference towards the consumption good. We assume that the consumer can always choose *not* to purchase, thus obtaining a reservation level of utility equal to zero. In this case, we let $q = t = 0$.

Suppose that the total payment depends on the price per unit purchased p and on a non-negative fixed fee f , so that the consumer's maximization problem is

$$\max_{q, t} u(q, t) \quad \text{subject to} \quad t = pq + f \quad \text{and} \quad q \geq 0$$

Substituting for t into the utility function and recalling (4), the above problem is equivalent to

$$\max_{q \geq 0} (\theta - p)q - \left(\frac{\theta}{2} \right) q^2 - f \quad (5)$$

Except for the constant f , (5) is a standard problem with $a = \theta - p$ and $b = \theta/2$.

Since $f \geq 0$, the solution to (5) is $q^* = 0$ if $p \geq \theta$. When $p < \theta$, by the usual argument the consumer chooses

$$q^* = \frac{\theta - p}{\theta} = 1 - \frac{p}{\theta} \quad (6)$$

if the maximum value in (5) is non-negative, i.e. if $S^* - f \geq 0$, where $S^* = \theta v(q^*) - pq^*$ denotes the *consumer surplus*. When $S^* < f$, the fixed fee is too high and the consumer does not purchase, i.e. $q^* = 0$.

Summing up, the optimal choice for the consumer is $q^* = 0$ if either $p \geq \theta$ or $p < \theta$ and

$f > S^*$. If instead $p < \theta$ and $f \leq S^*$, the consumer purchases $q^* = 1 - p/\theta$.

Example 3 (Portfolio choice with mean-variance preferences). A consumer has to allocate a given amount of money m between an asset which yields a safe return $r \geq 0$ and a risky asset, whose random return is \tilde{R} , with mean μ_R and variance σ_R . For the risky asset to be purchased, we require $\mu_R > r$.

A portfolio is an allocation (x, y) such that $m = x + y$, where y denotes the amount invested in the safe asset and x that invested in the risky asset. Suppose the consumer chooses the portfolio according to the mean-variance criterion $\mu - \delta\sigma$, in which $\delta > 0$ is a parameter capturing her risk-aversion, and μ and σ are, respectively, the mean and the variance of the portfolio return.

Let $\tilde{r} = y(1+r) + x(1+\tilde{R})$ be the portfolio return. Since $y = m - x$, \tilde{r} can be rewritten as $m(1+r) + x(\tilde{R} - r)$. The mean and the variance of the portfolio return are, therefore, equal to $\mu = m(1+r) + x(\mu_R - r)$ and $\sigma = x\sigma_R$.

If we do not impose non-negativity constraints on x and y , the portfolio problem is

$$\max_x m(1+r) + (\mu_R - r)x - \delta\sigma_R x^2, \quad (7)$$

which is equivalent to the standard problem with $a = (\mu_R - r)$ and $b = \delta\sigma_R$, since $m(1+r) > 0$. The amount of money invested in the risky asset is therefore $x^* = (\mu_R - r)/2\delta\sigma_R$.

Example 4 (Profit maximization in a competitive market). Consider a firm operating in a competitive market for the commodity q . The firm takes the commodity price p as given and seeks to maximize profits. The cost function is $C(q) = \alpha + \beta q + \gamma q^2$, with $\alpha, \beta, \gamma > 0$ if q is positive and zero otherwise, i.e. $C(0) = 0$. The profit maximization problem is $\max_{q \geq 0} pq - C(q)$, which corresponds to the following standard problem

$$\max_{q \geq 0} (p - \beta)q - \gamma q^2 - \alpha \quad (8)$$

Since $\alpha > 0$, the solution to (8) is $q^* = 0$ if $p \leq \beta$. When $p > \beta$, by the usual argument the firm chooses $q^* = (p - \beta)/2\gamma$ if the maximum value in (8) is non-negative, i.e. if $P^* - \alpha \geq 0$, where $P^* = pq^* - \beta q^* - \gamma(q^*)^2$ is the *producer surplus*. When $P^* < \alpha$, the (non-sunk) fixed

cost α is too high and the firm chooses $q^* = 0$.

Observe that $q^* = (p - \beta)/2\gamma$ is the solution to (8) also if either $\alpha = 0$ (no fixed costs) or $C(0) = \alpha$ (sunk fixed costs). When there are sunk costs, while it is clear that $P^* > 0$, it may well be possible that $P^* - \alpha < 0$, i.e. the firm is active even though profits are negative.

Example 5 (Tax revenue in partial equilibrium). Consider a competitive market for a given commodity. Let the demand be $q^d(p + x) = \alpha - \beta(p + x)$, with $\alpha, \beta > 0$ and $\alpha - \beta x > 0$, in which $x \geq 0$ is the unit tax levied on the commodity and p the market price for the commodity. Supply is $q^s(p) = \delta p$, with $\delta > 0$. Supply equals demand, i.e. $q^d(p + x) = q^s(p)$, when $p = (\alpha - \beta x)/(\beta + \delta)$. At equilibrium, the quantity exchanged on the market is $q = a - bx$, with $a = \alpha\delta/(\beta + \delta)$ and $b = \beta\delta/(\beta + \delta)$

If a public authority wants to maximize revenues from taxes xq , she solves the following problem

$$\max_x ax - bx^2, \quad \text{subject to } 0 \leq x \leq c. \quad (9)$$

with $c = \alpha/\beta$. Within the interval of feasible unit tax, (9) is a standard problem. Since the standard solution $x^* = a/2b = \alpha/2\beta$ is feasible, we conclude that this is indeed the optimal unit tax.

4.2 Strategic interactions

We now propose some economic applications of solution concepts for strategic interactions, commonly presented in game theory and microeconomics courses. We start with the duopolistic models á la Cournot and á la Bertrand and the first-price sealed-bid auction, which are classic applications of Nash Equilibrium. We then present the Stackelberg model to illustrate Subgame Perfect Nash Equilibrium.

Example 6 (Cournot duopoly). Consider a duopolistic market, in which firm 1 and 2 seek to maximize profits by selling a homogeneous good, and let the aggregate demand be $Q = \alpha - \beta P$ if $0 \leq P < \alpha/\beta$ and $Q = 0$ if $P \geq \alpha/\beta$. To analyze this problem, it is

customary to work with an inverse demand function

$$P(Q) = \begin{cases} \frac{\alpha}{\beta} - \frac{1}{\beta}Q & \text{if } 0 \leq Q < \alpha \\ 0 & \text{if } Q \geq \alpha \end{cases} \quad (10)$$

in which $Q = q_1 + q_2$ with $q_i \geq 0$ being the production of firm $i = 1, 2$. Firms are identical in the cost structure and compete by simultaneous choices on their individual production. Let $C(q_i) = \gamma q_i$, with $0 \leq \gamma < \alpha/\beta$, be the total cost of firm $i = 1, 2$.³ Firm's profits are given by

$$P(Q)q_i - C(q_i) = \begin{cases} \left(\frac{\alpha}{\beta} - \frac{1}{\beta}(q_i + q_j) \right) q_i - \gamma q_i & \text{if } 0 \leq q_i < \alpha - q_j \\ -\gamma q_i & \text{if } q_i \geq \alpha - q_j \end{cases} \quad (11)$$

for $i = j = 1, 2$ and $j \neq i$.

Firms determine their production simultaneously: each firm takes as given the quantity that her opponent decides to produce and defines her best response. Let $a_j = (\alpha - q_j) / \beta - \gamma$, $b = 1/\beta$ and $c_j = \alpha - q_j$. The profit maximization problem of firm i can be rewritten in the standard form as follows

$$\max_{q_i} a_j q_i - b q_i^2, \quad \text{subject to } 0 \leq q_i \leq c_j \quad (12)$$

in which, of course, the parameters a_j and c_j depend on q_j . Since $b > 0$, when $a_j < 0$, i.e. $(\alpha - q_j)/\beta < \gamma$, the residual demand for firm i is too low and it is optimal to set $q_i = 0$. If instead $a_j \geq 0$, firm i 's optimal production is

$$q_i(q_j) = \frac{a_j}{2b} = \frac{\alpha - \beta\gamma}{2} - \frac{q_j}{2}, \quad (13)$$

which is the best reply of firm i to the production choice of firm j . We just have to check that $q_i(q_j)$ is feasible in problem (12). To do so, observe that since $c_j/\beta \geq \gamma$, $2c_j/\beta \geq c_j/\beta + \gamma = a_j + 2\gamma > a_j$ hence $c_j \geq a_j/2b = q_i(q_j)$, which makes the solution feasible.⁴

³The quadratic cost case, $C(q_i) = \gamma q_i^2$, can be dealt with using the same methodology.

⁴For an analysis of the Cournot model with a methodology close in spirit to ours, see Dufwenberg 2001.

Example 7 (Bertrand duopoly with heterogenous goods). Consider now a duopolistic market, in which firm 1 and 2 produce two goods that are imperfect substitutes and seek to maximize profits by competing on prices. The demand function for firm $i = 1, 2$ is

$$q_i(p_i, p_j) = \begin{cases} \alpha - p_i + \beta p_j & \text{if } 0 \leq p_i < \alpha + \beta p_j \\ 0 & \text{otherwise} \end{cases}$$

in which $p_i \geq 0$ ($p_j \geq 0$) is the price of the good produced by firm i (j) and $\beta > 0$ captures the degree of substitutability between goods. Firms have identical cost functions $C(q_i) = \gamma q_i$ with $0 \leq \gamma < \alpha$ for $i = 1, 2$. Therefore, profits of firm i are

$$(p_i - \gamma) q_i(p_i, p_j) = (p_i - \gamma) (\alpha - p_i + \beta p_j) ,$$

and its profit maximization problem can be rearranged into

$$\max_{p_i} (\alpha + \beta p_j + \gamma) p_i - p_i^2 - (\alpha + \beta p_j) \gamma, \quad \text{subject to } 0 \leq p_i < \alpha + \beta p_j, \quad (14)$$

As usual, this is a standard problem with $a_j = \alpha + \beta p_j + \gamma$ and $b = 1$. Since $a_j > 0$ because of our assumptions, the best reply of firm i to firm j 's decision is

$$p_i(p_j) = \frac{a_j}{2b} = \frac{\alpha + \beta p_j + \gamma}{2}$$

It is easy to show that $p_i(p_j)$ is feasible, since $\gamma < \alpha < \alpha + \beta p_j$, and guarantees to firm i positive profits.

Example 8 (First price, sealed bid auction). Suppose there are two buyers $i = 1, 2$ who wish to bid for an indivisible good. Buyer's i valuation of the good is denoted by r_i and it is uniformly distributed on $[0, 1]$. Each buyer submits her bid x_i in a sealed envelope and the good is assigned to the highest bidder.

Assume it is common knowledge that $x_2 = a_2 r_2$ for some $a_2 > 0$, i.e. that x_2 is proportional to r_2 . Buyer 1 obtains the good if $x_2 \leq x_1$ and gets payoff $r_1 - x_1$. Observe that the highest possible bid for buyer 2 is a_2 , and that $x_2 \leq x_1$ is equivalent to $r_2 \leq x_1/a_2$,

which implies that $\Pr(x_2 \leq x_1) = \Pr(r_2 \leq x_1/a_2) = x_1/a_2$. Hence, the expected payoff for buyer 1 is

$$(r_1 - x_1) \Pr(x_2 < x_1) = \left(\frac{1}{a_2} \right) (r_1 x_1 - x_1^2).$$

Since $a_2 > 0$, buyer 1's optimal bid is the solution to the standard problem

$$\max_{x_1} r_1 x_1 - x_1^2, \quad \text{subject to } 0 \leq x_1 \leq a_2,$$

which is $x_1^* = r_1/2$ within the feasible interval; otherwise $x_1^* = a_2$. By symmetry, buyer 2's optimal bid has an identical structure.

Example 9 (Stackelberg duopoly). Consider again a duopolistic market, in which two firms seek to maximize profits by selling a homogeneous good on a market in which the aggregate inverse demand is given by (10). As in Example 6, $Q = q_1 + q_2$ with $q_i \geq 0$ being the individual production of firm $i = 1, 2$ and firms have identical cost structures $C(q_i) = \gamma q_i$ for every $i = 1, 2$. The game changes in that firms decide on their production sequentially. Let firm i be the *leader* and firm j the *follower*. The follower determines her production taking as given the quantity selected by the leader; the leader instead determines her optimal production anticipating the optimal choice of the follower.

It follows that the problem of the follower is identical to the one analyzed in the Cournot duopoly; in particular, the profits of the follower are still given by (11). As for the leader, the profits are

$$P(Q)q_i - C(q_i) = \left(\frac{\alpha}{\beta} - \frac{1}{\beta} (q_i + q_j(q_i)) \right) q_i - \gamma q_i \quad (15)$$

where $q_j(q_i)$ is the follower's best reply given in (13). By substituting $q_j(q_i)$ in (15), the leader's problem has the standard form

$$\max_{q_i} \left(\frac{\alpha}{\beta} - \gamma \right) q_i - \frac{1}{\beta} q_i^2 \quad \text{subject to } 0 \leq q_i < \alpha + \beta\gamma, \quad (16)$$

in which $a = \alpha/\beta - \gamma$, $b = 1/\beta$ and $c = \alpha + \beta\gamma$. It follows that within the range of feasible quantities, the optimal production of the leader is $q_i = a/2b = (\alpha - \gamma\beta)/2$ and of the

follower $q_j = (\alpha - \gamma\beta)/4$.⁵

4.3 Pricing strategies in a monopolistic market

In this section, we consider the problem of a monopolist selling a good on the market and compare alternative pricing rules. We first consider the case of a single consumer and then extend to the case of several consumers with heterogeneous preferences, which may be observable or not.⁶ Let the consumer preferences be given by the utility function $u(q, t) = \theta v(q) - t$, in which $v(q) = q - q^2/2$ as in Example 2. Recall that the consumer demand is given by

$$q(p) = 1 - \frac{p}{\theta} \quad (17)$$

and the consumer surplus by $S(p) = \theta v(q(p)) - pq(p) = (\theta - p)^2/2\theta$.

We develop the discussion considering three main pricing strategies: a non-linear tariff $t(q)$, a two-part tariff $t = pq + f$ and linear tariff, $t = pq$.

4.3.1 Homogeneous consumer preferences

Example 10 (Non-linear tariff). Consider a monopolist with cost function $C(q) = \gamma q$, in which $0 < \gamma < \theta$. The monopolist takes the consumer preferences as given and chooses a pair (t, q) to maximize profits, which are given by $t - \gamma q$. Since the consumer can always reject the proposal, the monopolist must take into account the constraint

$$\theta \left(q - \frac{q^2}{2} \right) - t \geq 0$$

so that the consumer is indeed willing to trade. No restrictions on the (t, q) -space are imposed. Hence, the profit maximization problem for the monopolist is simply

$$\max_{t, q} t - \gamma q, \quad \text{subject to } \theta \left(q - \frac{q^2}{2} \right) - t \geq 0 \quad (18)$$

⁵Along the same lines of this example, it is easy to deal with the quadratic cost case $C(q_i) = \gamma q_i^2$.

⁶This section is inspired by Tirole 1988 and Bolton and Dewatripont 2005.

Given the structure of the problem, it is optimal for the monopolist to set the tariff equal to $t^* = \theta(q - q^2/2)$. Hence, substituting into the profit function, we can easily get the standard form that follows

$$\max_q (\theta - \gamma)q - \left(\frac{\theta}{2}\right)q^2 \quad (19)$$

in which the optimal quantity is given by $q^* = (\theta - \gamma)/\theta$.

(t^*, q^*) is a take-it-or-leave-it offer from the monopolist to the consumer and t^* is a non-linear tariff, which depends on the taste parameter θ . From the welfare perspective, (t^*, q^*) is first-best (Pareto) efficient, since the monopolist is eventually maximizing the social surplus. From the point of view of the distribution, the consumer does not get any surplus, which accrues entirely to the monopolist.

Example 11 (Two-part tariff). Consider now a monopolist, who uses a two-part tariff $t = pq + f$. That is, she sets a linear price per unit of product sold with the option of introducing a fixed fee. As clarified in Example 2, the consumer demand is positive if $p < \theta$ and $f \leq S(p)$, and it is zero if either $p \geq \theta$ or $p < \theta$ and $f > S(p)$. Hence, the relevant region to solve the monopolist problem is $p < \theta$ and $f \leq S(p)$. Let the profits be $pq(p) + f - \gamma q(p)$, the monopolist solves the following problem

$$\max_{p,f} pq(p) + f - \gamma q(p) \quad \text{subject to} \quad 0 \leq p < \theta, \quad f \leq S(p) \quad (20)$$

Since profits are increasing with f , it is optimal to set the fixed fee equal to $S(p)$. Hence, the problem is equivalent to $\max_p pq(p) + S(p) - \gamma q(p)$, which boils down to $\max_p \theta v(q(p)) - \gamma q(p)$ after substituting for the consumer surplus. Recalling that $v(q) = \theta(q - q^2/2)$, problem (20) can be rewritten in the standard form as follows

$$\max_p (\theta - \gamma)q(p) - \left(\frac{\theta}{2}\right)q(p)^2,$$

with $a = \theta - \gamma$ and $b = \theta/2$. In the above problem, the maximum value is equal to $(\theta - \gamma)^2/2\theta > 0$ and it is attained when $q(p) = a/2b = 1 - \gamma/\theta$. The best choice for the monopolist is therefore to choose a price p^* such that $q(p^*) = 1 - \gamma/\theta$ and set $f^* = S(p^*)$ to extract all the consumer's surplus. Given (17), the optimal unit price for the monopolist is equal

to marginal cost, $p^* = \gamma < \theta$.

If there is a single consumer, or several identical ones, the two-part tariff and the non-linear tariff achieve the same efficient allocation and guarantee to the monopolist the same surplus.

Example 12 (Linear tariff). Consider now the case in which the monopolist can only choose a linear price schedule, i.e. constant unit price and $f = 0$. Her profits are equal to $pq(p) - \gamma q(p)$ if $p < \theta$ and to zero otherwise. Substituting for the demand function,

$$pq(p) - \gamma q(p) = (p - \gamma) \left(1 - \frac{p}{\theta}\right) = \left(\frac{\theta + \gamma}{\theta}\right) p - \frac{p^2}{\theta} - \gamma,$$

and the profit maximization problem of the monopolist becomes

$$\max_p \left(\frac{\theta + \gamma}{\theta} \right) p - \left(\frac{1}{\theta} \right) p^2 - \gamma \quad (21)$$

Except for γ , (21) is a problem in the standard form with $a = (\theta + \gamma)/\theta$ and $b = 1/\theta$. Therefore, the candidate solution is $p^* = a/2b = (\theta + \gamma)/2 < \theta$ under our assumptions, with maximum profits equal to $(\theta + \gamma)^2/4\theta - \gamma$. Since $\gamma < \theta$, it is true that $(\theta - \gamma)^2 = \theta^2 + \gamma^2 - 2\theta\gamma > 0$, and $4\theta\gamma < \theta^2 + \gamma^2 + 2\theta\gamma = (\theta + \gamma)^2$, from which it follows that $(\theta + \gamma)^2/4\theta - \gamma > 0$ and profits are positive. At p^* the monopolist earns strictly positive profits, while for any $p \geq \theta$ profits would be zero. Therefore, the optimal choice for the monopolist is $p^* = (\theta + \gamma)/2$, i.e. a price higher than marginal cost.

4.3.2 Heterogeneous consumers with observable preferences

We extend our example to the case in which there are $i = 1, 2, \dots, I$ types of consumers, with $I \geq 2$. The proportion of each type is $\lambda_i > 0$, with $\sum_i \lambda_i = 1$. Types differ in the taste parameters θ_i , which are observable to the monopolist and such that $\theta_1 < \theta_2 < \dots < \theta_I$. Let the utility of type i be $u_i(q_i, t_i) = \theta_i v(q_i) - t_i$. We can directly extend (17) to this case by letting $q_i(p) = 1 - p/\theta_i$ be the demand of type i .

Let the monopolist's cost function be $C(q) = \gamma q$, with $\gamma < \theta_1$, to ensure that every type can be served by the monopolist. For the sake of clarity, let $1/\hat{\theta} = \sum_i \lambda_i/\theta_i$, so that $\hat{\theta}$ is

the (generalized) harmonic mean of the taste parameters, and $\hat{q}(p) = \sum_i \lambda_i q_i(p) = 1 - p/\hat{\theta}$ be the (average) aggregate demand.

Example 13 (Non-linear tariff or first-best of a screening model with two types). If types are observable, the problem of the monopolist is to find an optimal tariff-quantity pair (t_i, q_i) for every type of consumer, which amounts to solve

$$\begin{aligned} & \max_{t_i, q_i} \sum_i \lambda_i (t_i - \gamma q_i) \\ & \text{subject to } \theta_i \left(q_i - \frac{q_i^2}{2} \right) - t_i \geq 0 \quad \forall i = 1, \dots, I \end{aligned}$$

Observe that the monopolist problem is separable with respect to consumers' types and for every type it has the same structure as in (18). The optimal quantity proposed to each type will thus be given by $q_i^* = (\theta_i - \gamma) / \theta_i$.

With heterogeneous and observable preferences, in the two-part tariff case the monopolist chooses a unit price p_i and a fixed fee f_i for every type. Observe that the corresponding profit maximization problem is separable with respect to consumers' types, thus the analysis mirrors the one for the single consumer illustrated in Example 11. Hence, the monopolist optimally sets the unit price equal to marginal cost $p_i^* = \gamma$ and the fixed fee $f_i^* = S_i(\gamma)$ for every i extracting all the surplus. When preferences are observable, the two-part tariff and the non-linear tariff achieve the same optimal allocation.

If the monopolist is constrained to set $f_i = 0$ for every i , she can either choose a unique price for all types or set different unit prices across types, since these are observable. The first case corresponds to the textbook monopoly problem, the second one to third-degree price discrimination.

Example 14 (Linear tariff or the classical monopoly problem). Suppose the monopolist sets a single price on the market, so that profits are given by

$$(p - \gamma) \hat{q}(p) = (p - \gamma) \sum_i \lambda_i q_i(p)$$

Recalling that $\hat{q}(p) = 1 - p/\hat{\theta}$, the profit maximization problem can be rewritten as

$$\max_p \left(\frac{\hat{\theta} + \gamma}{\hat{\theta}} \right) p - \left(\frac{1}{\hat{\theta}} \right) p^2 - \gamma \quad (23)$$

Except for γ , (23) is a problem in the standard form with $a = (\hat{\theta} + \gamma)/\hat{\theta}$ and $b = 1/\hat{\theta}$. The maximum value of profits is $(\hat{\theta} + \gamma)^2/4\hat{\theta} - \gamma$. A well-known property of (generalized) means implies that $\theta_1 < \hat{\theta}$, therefore, under our assumption, $\gamma < \theta_1 < \hat{\theta}$. By the same reasoning as in Example 12, $\gamma < \hat{\theta}$ implies $\gamma < (\hat{\theta} + \gamma)^2/4\hat{\theta}$, i.e. profits are positive. Therefore, $p^* = a/2b = (\hat{\theta} + \gamma)/2$ is indeed the solution to problem (23).

Observe that when the monopolist sets a unique price for every type of consumer, the solution implies bunching at the optimum.

Example 15 (Third-degree price discrimination). Suppose now that, absent the fixed fee $f_i = 0$, the monopolist sets personalized prices p_i for every type, so that her profits are given by

$$\sum_i \lambda_i (p_i - \gamma) q_i(p_i)$$

In this case, he will solve the following problem for every i

$$\max_{p_i} \left(\frac{\theta_i + \gamma}{\theta_i} \right) p_i - \left(\frac{1}{\theta_i} \right) p_i^2 - \gamma$$

in which we used $q_i(p_i) = 1 - p_i/\theta_i$. The problem above is identical to (21), hence its solution is $p_i^* = (\theta_i + \gamma)/2$ for every i . It is optimal for the monopolist to set personalized prices which are increasing in the taste parameter θ_i .

4.3.3 Heterogeneous consumers with non-observable preferences

Suppose that consumers' preferences are not observable, i.e. the monopolist cannot verify the taste parameter θ_i . For the sake of simplicity, assume that there are only two types so that $\theta_1 < \theta_2$. Let the cost function be $C(q) = \gamma q$ with $0 < \gamma < \theta_1$ and consider the case in which the monopolist serves both types of consumers.

Example 16 (Non-linear tariff). Let $\lambda_1 = \lambda$ be the proportion of type-1 consumers. Consider the optimal tariff-quantity pair which discriminates among types. The monopolist's profit maximization problem can be written as follows

$$\begin{aligned} \max_{t_1, q_1, t_2, q_2} \quad & \lambda(t_1 - \gamma q_1) + (1 - \lambda)(t_2 - \gamma q_2) \\ \text{subject to} \quad & \theta_1 \left(q_1 - \frac{q_1^2}{2} \right) - t_1 \geq 0 \end{aligned} \tag{IR1}$$

$$\theta_2 \left(q_2 - \frac{q_2^2}{2} \right) - t_2 \geq 0 \tag{IR2}$$

$$\theta_1 \left(q_1 - \frac{q_1^2}{2} \right) - t_1 \geq \theta_1 \left(q_2 - \frac{q_2^2}{2} \right) - t_2 \tag{IC1}$$

$$\theta_2 \left(q_2 - \frac{q_2^2}{2} \right) - t_2 \geq \theta_2 \left(q_1 - \frac{q_1^2}{2} \right) - t_1 \tag{IC2}$$

in which (IR1, IR2) are the participation constraints and (IC1, IC2) the incentive compatibility constraints for every type. Standard results on screening models imply that at the solution (IR1) and (IC2) bind, so that $t_1 = \theta_1 \left(q_1 - \frac{q_1^2}{2} \right)$ and $t_2 = \theta_2 \left(q_2 - \frac{q_2^2}{2} \right) - \Delta\theta \left(q_1 - \frac{q_1^2}{2} \right)$, with $\Delta\theta = (\theta_2 - \theta_1)$, and that $q_2^* = (\theta_2 - \gamma)/\theta_2$ which corresponds to the efficient quantity for type 2.⁷

By substituting into the expected profit function for the monopolist, we get

$$\max_{q_1} \lambda \left(\theta_1 \left(q_1 - \frac{q_1^2}{2} \right) - \gamma q_1 \right) + (1 - \lambda) \left(\theta_2 \left(q_2^* - \frac{(q_2^*)^2}{2} \right) - \gamma q_2^* - \Delta\theta \left(q_1 - \frac{q_1^2}{2} \right) \right),$$

which is equivalent to

$$\max_{q_1} (\lambda(\theta_1 - \gamma) - (1 - \lambda)\Delta\theta) q_1 - (\lambda\theta_1 - (1 - \lambda)\Delta\theta) \frac{q_1^2}{2} \tag{25}$$

Problem (25) is our standard problem in which $a = \lambda(\theta_1 - \gamma) - (1 - \lambda)\Delta\theta$ and $b = (\lambda\theta_1 - (1 - \lambda)\Delta\theta)/2$. Its solution is given by

$$q_1^* = \frac{\lambda(\theta_1 - \gamma) - (1 - \lambda)\Delta\theta}{\lambda\theta_1 - (1 - \lambda)\Delta\theta}$$

⁷See, for example, Salanié 2005 or Bolton and Dewatripont 2005.

The solution identified corresponds to the second-best (Pareto) efficient allocation since the taste parameters of the different consumers are not observable. In this case, the monopolist faces a trade-off between reducing the information rent of the type θ_2 by lowering q_1^* at the cost of offering to type 1 an inefficient quantity.

Example 17 (Second-degree price discrimination). Consider now the case in which the monopolist designs a two-part tariff, i.e. chooses p and f , to maximize profits. Since $\theta_2 > \theta_1$ and $S_2(p) > S_1(p)$ when $p < \theta_1$ ⁸, the monopolist will optimally set $f^* = S_1(p)$ in order to maximize revenues from the fixed fee while serving both types of consumers. Therefore, profits are equal to

$$\sum_i \lambda_i (pq_i(p) - \gamma q_i(p) + S_1(p)) = (p - \gamma) \hat{q}(p) + S_1(p)$$

Simple manipulations transform the profit maximization problem into

$$\max_p \left(\frac{\gamma}{\hat{\theta}} \right) p - \left(\frac{2\theta_1 - \hat{\theta}}{2\theta_1 \hat{\theta}} \right) p^2 + \left(\frac{\theta_1 - 2\gamma}{2} \right) \quad (26)$$

Except for the constant term, (26) is a problem in the standard form with $a = \gamma/\hat{\theta}$ and $b = (2\theta_1 - \hat{\theta})/2\theta_1 \hat{\theta}$. If $2\theta_1 - \hat{\theta} > 0$, the optimal price and quantity, as well as profits are positive. Therefore, $p^* = a/2b = \gamma\theta_1/(2\theta_1 - \hat{\theta})$ is the solution to problem (26).

Finally, if the monopolist uses a linear tariff when types are not observable, her problem is identical to (23), hence there will be bunching at the optimum.

References

- [1] David Besanko and Ronald R. Braeutigam (2014), *Microeconomics*, John Wiley & Sons, London, UK.
- [2] Patrick Bolton and Mathias Dewatripont (2005), *Contract Theory*, The MIT Press, Cambridge, MA.

⁸Since $(\theta_2 - p)/\theta_2 > (\theta_1 - p)/\theta_1$ and $\theta_2 - p > \theta_1 - p > 0$, it follows that $S_2(p) = (\theta_2 - p)^2/2\theta_2 > (\theta_1 - p)^2/2\theta_1 = S_1(p)$.

- [3] Martin Dufwenberg (2001), *Teaching Cournot Without Derivatives*, Journal of Economic Education, 32(1), 36-40.
- [4] Roger B. Nelsen (1993), *Proofs Without Words*, Mathematical Association of America, Washington DC.
- [5] Ivan Niven (1981), *Maxima and Minima Without Calculus*, Mathematical Association of America, Washington DC.
- [6] Bernard Salanie (2005), *The Economics of Contracts. A Primer*, The MIT Press, Cambridge, MA.
- [7] Jean Tirole (1988), *The Theory of Industrial Organization*, The MIT Press, Cambridge, MA.
- [8] Hal R. Varian (2010), *Intermediate Microeconomics*, W.W. Norton & Company, New York, NY.