budget constraint

1) $c_1 + c_2/(1+r) <= w_1 + w_2/(1+r)$

2) maximize $U = u(c_1) + u(c_2)/(1+d)$ subject to 1
c_1,c_2

d is the subjective rate of discounting of future happiness.

3) $c_2 = w_1(1+r) + w_2 - c_1(1+r)$

so

4) $U = u(c_1) + [u(w1(1+r) + w_2 - c_1(1+r))]/(1+d)$

Equation 4 gives the first order condition equation 5

5) $0 = u'(c_1) - u'(c_2)(1+r)/(1+d)$

Equation 5 states that $u'(c_1) = u'(c_2)(1+r)/(1+d)$, it is conventional to write $u'(c_1) = u'(c_2)(1+r)/(1+d) =$ lambda, and call lambda a Lagrange multiplier. Lagrange generalized the trivial derivation above, but basically he did the same thing.

6)

$\max_{c_1,c_2} u(c_1) + u(c_2)/(1+d) -\lambda[c_1 + c_2/(1+r) -w_1 -w_2/(1+r)]$

Now assume that the consumer lives forever, consumes c_t in period t and earns w_t in period t. The consumer chooses c_i in each period to maximize the intertemporal utility function

$$7)\ U = \sum_{t=1}^{\infty} u(c_t)/(1+d)^t$$

subject to the budget constraint

$$8)\ \sum_{t=1}^{\infty} (c_t - w_t)/(1+r)^t <= 0$$

By noting the analogy with equation 6 and appealing to Lagrange you might convince yourself that maximizing 7 subject to 8 is equivalent to unconstrained maximization of equation 9

$$9)\ \max \sum_{t=1}^{\infty} [\ u(c_t)/(1+d)^t] - \lambda \sum_{t=1}^{\infty} [(c_t-w_t)/(1+r)^t\}$$

so

$$10)\ u'(c_t) = \lambda(1+d)^t/(1+r)^t$$

$$11)\ u'(c_{t+1}) = u'(c_t)[(1+d)/(1+r)]$$

$$12)\ \text{MAXIMIZE } E_t\{ \sum^{\infty} [u(c_s)/(1+d)^{s-t}]\}$$

$c_s, s>=t \quad\quad s=t$

subject to the constraint

$$13) \quad \sum_{s=t}^{\infty} (c_s - w_s)/(1+r)^{s-t} <= S_t$$

Where S_t is financial wealth at time t.

$$14) \; E_s[u'(c_{s+1})/u'(c_s)] = (1+d)/(1+r)$$

Assume

$$u(c) = [c^{1-\sigma}]1/(1-\sigma)$$

which implies that $u'(c) = c^{-\sigma}$ and turns equation 14 into the highly useful equation 15

$$15) \; E_s(c_{s+1}/c_s)^{-\sigma} = (1+d)/(1+r)$$

Which making a bold approximation for logarithms implies

$$16) \; E_s[\log(c_{s+1}) - \log(c_s)] = (r - d)/\sigma$$

which can be written in the usable form

$$17) \; \log(c_{s+1}) = \log(c_s) + (r-d)/6 + e_s$$

where e_s is a disturbance term which is according to the rational expectations hypothesis uncorrelated with any lagged information. Equation 17 is testable, it implies that if log consumption is regressed on lagged log consumption and on lagged variables, the coefficient on log consumption will be one and the other coefficients will be zero.

Put briefly most of the tests have rejected the restrictions implied by equation 17. This implies that one of the many assumptions made in deriving equation

17 must be false.

First it is assumed that consumers are expected intertemporal utility maximizers.
Second it is assumed that they have rational expectations. Third it is assumed that consumers are free to borrow and to borrow at the same interest rate they earn when they save. That is it is assumed that they are not liquidity constrained. A form of the utility function is assumed. It is easy to check that results do not depend on the particular form of the one period utility function u by using different assumptions.

More importantly it was assumed that the intertemporal utility function is time separable.

Equation 17 was derived for a single consumer. It is usually tested with aggregate data. An additional assumption is made -- that aggregate consumption behaves as if there were a single representative consumer. Nonetheless the model has also been tested and rejected with data on individual consumption.

In this lecture I have assumed that the real interest rate r is known and constant. This assumption can be relaxed by testing whether lagged information helps predict changes in consumption only because it helps predict real interest rates. The modified hypothesis is also rejected by the data.

This leaves the following possible explanations for the rejection of the predictions of the life-cycle permanent income hypothesis -- that consumers are not utility maximizers, that they do not have rational expectations, that the utility function is not time separable and that they are not free to borrow.

Appendix:

Between equation 13 and equation 14 there should be an argument as follows: If c_t is optimal and the plan which gives c_s $s>t$ as a function of new information is optimal then there is no other c'_t and new plan c'_s $s > t$ which improves expected utility. In particular there is no change in which $c'_s = c_s$ for all $s > t+1$ which increases expected utility. Given the budget constraint this is equivilant to saying there is no δ such that expected utility is increased if c'_t

$= c_t + \delta$ and $c'_{t+1} = c_{t+1} - (1+r)\delta$. Note c_{t+1} is not known at t it depends on things the consumer learns after t, but the rational consumer plans c_{t+1} as a function of this new information and can imagine changing the plan by consuming $(1+r)\delta$ less in every case. When considering modified consumption plans of this type it is clear that the FOC is that the derivative of expected utility with respect to δ is zero at $\delta = 0$ or

14) $E_i[u'(c_{i+1})/u'(c_i)] = (1+d)/(1+r)$

Now the main point. I assume that the budget constraint must be satisfied with certainty. Dissapointingly low w must be balanced by low c. This may not always be possible if the required c is negative. Then the consumer goes bankrupt. Creditors would not loan at the safe rate r to a consumer who might go bankrupt. For creditors to be willing to loan any amount the consumer wants to borrow at the same interest rate consumers receive on savings it is necessary that consumers do not want to risk bankruptcy -- that they choose to borrow only so much that there consumption is certainly strictly positive in each period. Consumers will choose to do this if the consequences of zero or extremely low consumption are sufficiently horrible, that is, if the slope of the u(c) goes to infinity sufficiently quickly as c goes to 0.

Returning to the argument behind equation 14, for the FOC to hold it is necessary that the consumer not choose to be at a corner. I argue that the derivative of expected utility with respect to δ must be zero at $\delta = 0$. Otherwise it would be possible to increase expected utility for δ slightly positive or slightly negative. For this argument to be valid, slightly positive and slightly negative δ must be feasible. In other words c_t must be positive making it possible to reduce c_t by a small amount and c_{t+1} must be a random variable bounded away from zero (certainly greater than or equal to some positive amount) making it certain that c_{t+1} can be reduced by $(1+r)\delta$ for some positive δ. If this is not always true with certainty for every t, equation 14 is not valid.

The assumption that consumers are free to borrow any amount at the same real interest rate, and the assumption that lenders have rational expectations together require and imply that consumers will never choose to risk bankruptcy which sould imply equation 14. If consumers are willing to risk bankruptcy (as we certainly are) creditors will charge different interest rates depending on the risk of bankruptcy or refuse to lend at all (as they certainly do). The possibility that consumers might choose to risk banrkuptcy not only implies that we sometimes violate our budget constraint, but also implies that rational creditors are not willing to lend us any amount that we wish to borrow at the same interest rate. This might explain why equation 14 does not hold in practice and is as noted the most popular proposed explanation.