

Exercises 5th Week: Solution

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Exercise 2

Let X be a random variable with p.d.f.:

$$f(x|\theta) = \frac{\exp(-(x-\theta))}{(1+\exp(-(x-\theta)))^2} \quad -\infty < x < \infty \quad -\infty < \theta < \infty$$

Use the pivotal method to verify that if $0 < \alpha_1 < 0.5$ and $0 < \alpha_2 < 0.5$, then

$$\left(X - \log\left(\frac{1-\alpha_2}{\alpha_2}\right), X - \log\left(\frac{\alpha_1}{1-\alpha_1}\right) \right)$$

is a confidence interval for θ with coverage probability $1 - (\alpha_1 + \alpha_2)$.

Solution Exercise 2

- Pivotal Variable : $Y = X - \theta$
- Distribution of Y

$$\begin{aligned} f_Y(y) &= \frac{\exp(-y)}{(1+\exp(-y))^2} \quad -\infty < y < \infty \\ F_Y(y) = P(Y \leq y) &= \int_{-\infty}^y \frac{\exp(-t)}{(1+\exp(-t))^2} dt = \frac{1}{1+\exp(-y)} \end{aligned}$$

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$$\begin{aligned} &P\left(X - \log\left(\frac{1-\alpha_2}{\alpha_2}\right) < \theta \leq X - \log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right) \\ &= P\left(\log\left(\frac{\alpha_1}{1-\alpha_1}\right) \leq Y < \log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right) \\ &= P\left(Y < \log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right) - P\left(Y < \log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right) \\ &= \frac{1}{1+\exp\left(-\log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right)} - \frac{1}{1+\exp\left(-\log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right)} \\ &= \frac{1}{1+\frac{\alpha_2}{1-\alpha_2}} - \frac{1}{1+\frac{1-\alpha_1}{\alpha_1}} = 1 - \alpha_1 - \alpha_2 \end{aligned}$$

Exercise 3

Compare the asymptotic efficiency of method of moments estimator of parameter α of the Pareto distribution with maximum likelihood estimator (assume x_m known). Remind that a r.v. X with a Pareto distribution has the following density:

$$f(x; \alpha, x_m) = \alpha x_m^\alpha x^{-(\alpha+1)} \quad \text{for } x \geq x_m$$

Solution: Exercise 3

Observe that the joint pdf of $X = (X_1, \dots, X_n)$

$$\begin{aligned} f(x; \alpha, x_m) &= \prod_{i=1}^n \frac{\alpha x_m^\alpha}{x_i^{\alpha+1}} \\ &= \frac{\alpha^n x_m^{n\alpha}}{\prod_{i=1}^n x_i^{\alpha+1}} \\ &= g(t, \alpha) h(x) \end{aligned}$$

where $t = \prod_{i=1}^n x_i$ $g(t, \alpha) = c\alpha^n x_m^{n\alpha} t^{-(\alpha+1)}$ and $h(x) = 1$.

By the factorization theorem: $T(X) = \prod_{i=1}^n X_i$ is a sufficient statistics for α

The log-likelihood function is

$$\log L(\alpha, x_m) = n \log \alpha + n \alpha \log(x_m) - (\alpha + 1) \sum_{i=1}^n \log x_i$$

$$\begin{aligned} \frac{\delta \log L(\alpha, x_m)}{\delta \alpha} &= \frac{n}{\alpha} + n \log(x_m) - \sum_{i=1}^n \log x_i \\ \frac{\delta^2 \log L(\alpha, x_m)}{\delta \alpha^2} &= -\frac{n}{\alpha^2} \end{aligned}$$

Solving for $\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} = 0$, since $\frac{\delta^2 \log L(\alpha, x_m)}{\delta \alpha^2} < 0$ the mle of α is given by

$$\begin{aligned} \hat{\alpha}_{MLE} &= \frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} \\ \text{Var}(\hat{\alpha}_{MLE}) &= (nI(\alpha))^{-1} = \frac{\alpha^2}{n} \end{aligned}$$

Compute by yourself $E(X)$ and $\text{Var}(X)$

$$\begin{aligned} E(X) &= \frac{\alpha x_m}{\alpha - 1} \quad \alpha > 1 \\ \text{Var}(X) &= \frac{\alpha x_m^2}{(\alpha - 1)^2 (\alpha - 2)} \quad \alpha > 2 \end{aligned}$$

Equating first population moment with first sample moment, we find α_{MOM}

$$\begin{aligned} E(X) &= \bar{X} \\ \frac{\alpha x_m}{\alpha - 1} &= \bar{X} \\ \hat{\alpha}_{MOM} &= \frac{\bar{X}}{\bar{X} - x_m} \end{aligned}$$

Distribution of $\hat{\alpha}_{MOM}$ can be found by delta method, REMIND:

Let Y_n , $n = 1, 2, \dots$, be a sequence of random variables such that

$$\sqrt{n}(Y_n - \theta) \rightarrow_d N(0, \sigma^2)$$

Furthermore let $g(\cdot)$ be a twice differentiable real function defined on R such that $g'(\theta) \neq 0$. Then

$$\sqrt{n}(g(Y_n) - g(\theta)) \rightarrow_d N(0, (g'(\theta))^2 \sigma^2)$$

- $Y_n = \bar{X}$
- $\theta = E(\bar{X}) = \frac{\alpha x_m}{\alpha - 1}$
- $\sigma^2 = Var(\bar{X}) = \frac{1}{n} \frac{\alpha x_m^2}{(\alpha - 1)^2 (\alpha - 2)}$
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$$\begin{aligned} g(t) &= \frac{t}{t - x_m} \\ g'(t) &= \frac{-x_m}{(t - x_m)^2} \\ (g'(t))^2 &= \frac{x_m^2}{(t - x_m)^4} \\ (g'(\theta))^2 &= \frac{(\alpha - 1)^4}{x_m^2} \\ (g'(\theta))^2 \sigma^2 &= \frac{1}{n} \frac{\alpha(\alpha - 1)^2}{\alpha - 2} \end{aligned}$$

$$Var(\hat{\alpha}_{MOM}) = \frac{1}{n} \frac{\alpha(\alpha - 1)^2}{\alpha - 2}$$

Comparison of the asymptotic efficiency of the two estimators:

$$\frac{Var(\hat{\alpha}_{MOM})}{Var(\hat{\alpha}_{MLE})} = \frac{\frac{1}{n} \frac{\alpha(\alpha - 1)^2}{\alpha - 2}}{\frac{\alpha^2}{n}} = \frac{(\alpha - 1)^2}{\alpha(\alpha - 2)}$$

$$\begin{aligned}
\frac{(\alpha - 1)^2}{\alpha(\alpha - 2)} &> 1 \\
(\alpha - 1)^2 &> \alpha(\alpha - 2) \\
\alpha^2 - 2\alpha + 1 &> \alpha^2 - 2\alpha \\
1 &> 0 \\
&\text{always}
\end{aligned}$$

Exercise 4

The life times of the neon lamps in the Uni Mail rooms can be modelled by an exponential distribution, i.e. we suppose that $(X_1, \dots, X_n)$ are from the exponential distribution $Exp(\lambda)$ with probability distribution function:

$$f(x|\lambda) = \lambda \exp(-\lambda x) I_{(0, \infty)}(x)$$

We would like to construct point and interval estimates of the median survival time:

1. Compute the median $Me(\lambda)$ of the model,
2. Find $\hat{Me}(\lambda)_{MLE}$: the maximum likelihood estimator for $Me(\lambda)$, the median of this distribution,
3. Give an approximate (asymptotic) confidence interval for $Me(\lambda)$ based on $\hat{Me}(\lambda)_{MLE}$,
4. Show that $2\lambda \sum_i X_i$ is a pivotal statistic and give the exact confidence interval for $Me(\lambda)$.
5. Suppose that for a random sample of $n = 100$ neon lamps you observe $\bar{x} = 5$, compare the exact and approximate confidence interval and discuss results found.

Solution: Exercise 4

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2. Compute the median $Me(\lambda)$ of the model,

$$\begin{aligned}
 \int_0^{Me(\lambda)} f(x|\lambda) dx &= 0.5 \\
 \int_0^{Me(\lambda)} \lambda \exp(-\lambda x) dx &= \left|_0^{Me(\lambda)} -\exp(-\lambda x) \right. \\
 1 - \exp(-\lambda Me(\lambda)) &= 0.5 \\
 \exp(-\lambda Me(\lambda)) &= 0.5 \\
 Me(\lambda) &= \frac{1}{\lambda} \log(2)
 \end{aligned}$$

Median of an exponential distribution: $Median = \frac{\log(2)}{\lambda}$.

3. For invariance property of MLE: $\hat{Median}_{MLE} = \log(2)\bar{X}$ Find $\hat{Me}(\lambda)_{MLE}$: the maximum likelihood estimator for $Me(\lambda)$, the median of this distribution,

$$\begin{aligned}
 \hat{Me}_{MLE}(\lambda) &= \frac{1}{\bar{\lambda}} \log(2) \\
 \hat{\lambda}_{MLE} &= \frac{1}{\bar{X}} \\
 \hat{Me}(\lambda)_{MLE} &= \bar{X} \log(2)
 \end{aligned}$$

$$\left[\log(2)\bar{X} - z_{1-\alpha/2} \frac{(\log(2)\bar{X})^2}{n}; \log(2)\bar{X} + z_{1-\alpha/2} \frac{(\log(2)\bar{X})^2}{n} \right]$$