1^{st} Assignment Statistics due October 13^{th} 2025

Maura Mezzetti

Exercise 1

The battery life of a smartphone is uniformly distributed between 8 and 15 hours. Suppose you test 6 phones. What is the probability that the shortest observed lifetime exceeds 10 hours?

Exercise 2

Let $(X_1, X_2, \ldots, X_n)$ be independent random variables such that $X_i \sim Geometric(p)$, where the geometric distribution counts the number of failures before the first success:

$$P(X_i = k) = (1-p)^k p, \qquad k = 0, 1, 2, \ldots$$

Define the statistic $T(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{m} X_i \qquad m < n.$

Show that T is not sufficient for p using the conditional distribution method.

Exercise 3

The double exponential distribution is

$$f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}, \qquad -\infty < x < \infty$$

For an iid sample of size $n = 2m + 1$, find the mle of θ.

Exercise 4

Suppose survival times $(Y_1, Y_2, \ldots, Y_n)$ follow a Weibull distribution with pdf

$$f(y|\theta) = \theta y^{\theta-1} e^{-y^{\theta}}, \qquad y > 0,\, \theta > 0$$

Now assume right censoring at time $c > 0$, so the observed data are

$$X_i = min(Y_i,\, c)$$

(Imagine that Y_i is the true survival time of a patient, but the study only lasts for 5 years ($c = 5$). If a patient is still alive at the end of the study, we do not know their exact survival time, we only know that it is greater than 5. This situation corresponds to right censoring at 5 years)

1. Write down the likelihood function based on censored data.

Exercise 5

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as Beta distribution.

$$f(x|\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 \leq x \leq 1$$

Show that the following statistics

$$T = \frac{1}{n}\left(\sum_{i=1}^{n} log(1 - X_i)\right)^3$$

is a sufficient statistics for parameter β.

Exercise 6

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as

$$f(x;\theta) = \theta(1+x)^{-(\theta+1)} I_{(0,\infty)}(x) \qquad \theta > 1$$

1. Find the Method of Moments Estimator (MOM) and the Maximum Likelihood Estimator (MLE) of θ
2. Find the MLE of $1/\theta$

Exercise 7

An electrical circuit consists of three batteries $X_1,\ X_2,\ X_3$ connected in series to a lightbulb Y. We model the battery lifetimes as independent and identically distributed Exponential(λ) random variables (such that $E(X_i) = \lambda$). Our experiment to measure the lifetime of the lightbulb is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = min(X_1, X_2, X_3)$.

1. Determine the distribution of the random variable Y.
2. Compute $\hat{\lambda}_{MLE}$, the maximum likelihood estimator of λ.

3. Determine the mean square error of $\hat{\lambda}_{MLE}$.
4. Use the Cramer-Rao lower bound to prove that $\hat{\lambda}_{MLE}$ is the minimum variance unbiased estimator of λ.

Exercise 8

Suppose that income Y is distributed as a Pareto distribution:

$$f(y) = \alpha y^{-(\alpha+1)}$$

for $y \geq 1$ and $\alpha > 1$

- It is quite common to not observe all incomes, but only those that are higher than some threshold (so-called truncated variables). Assume that you observe only those individuals with an income greater than or equal to $9,000$\$, and their income is described by a random variable $Y^\star$. How is $Y^\star$ distributed?
- Your have a sample of size n drawn from the population of persons with incomes greater than or equal to $9,000$\$. What is the MLE of α?
- What is asymptotic distribution of the estimator in previous point?