Exam Statistics

10^{th} July 2026

Name..................

ID Number..................

This is a closed book exam. Answer all the following questions and solve all the following exercises.

TIME: 1 hour and 30 minutes

Exercise 1

(3 points) Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables. Let $f_\theta(x)$ and $F_\theta(x)$ be the density function and cumulative distribution function respectively. Let $\hat{\theta}$ be the MLE of θ, θ_0 be the true parameter, $L(\theta)$ be the likelihood function, $logL(\theta)$ be the loglikelihood function, and $I(\theta)$ be the Fisher information matrix.

1. To verify the hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, which of the following statements is NOT true:
   (a) Wald test requires only the unrestricted model to be estimated.
   (b) Score test requires only the restricted model to be estimated.
   (c) LR test requires both the restricted and unrestricted models to be estimated
   (d) The three statistical tests provide the same values in the observed sample
2. For each of the following statements indicate whether it is TRUE or FALSE:

An estimator $T = T(X)$ is a consistent estimator for θ if the distribution is asymptotically Gaussian	T	F
A statistic $T = T(X)$ is a sufficient statistic for θ if it can be computed without knowing the value of θ	T	F
The likelihood ratio is a random variable	T	F
The following function is a density function f(x)= $\begin{cases} 6x^2 - 1 & if x \in [0, 1] \\ 0 & otherwise \end{cases}$	T	F

Exercise 2

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as follows:

$$f(x;\theta) = \frac{(\theta-1)\,2^{\theta-1}}{x^{\theta}} \qquad x > 2 \quad \theta > 1$$

1. **(2 points)** Show that $\sum_i log(X_i)$ is a sufficient statistics for θ
2. **(2 points)** Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.
3. **(1 point)** Find $\hat{\theta}_{MOM}$ method of moment estimator (MOM) for θ.
4. **(2 points)** Compute the score function and the Fisher information.
5. **(1 point)** Specify asymptotic distribution of $\hat{\theta}_{MLE}$.
6. **(6 points)** Suppose form a random sample of 120 random variables you obtain $\sum_{i=1}^{120} log(x_i) = 115$. Fix $\alpha = 0.01$ and complete ALL of the following questions:
   (a) Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 5$ versus $H_1 : \theta \neq 5$, specify the distribution and verify the null hypothesis.
   (b) Find the Wald test statistic for testing $H_0 : \theta = 5$ versus $H_1 : \theta \neq 5$, specify the distribution and verify the null hypothesis.
   (c) Find the Score test statistic for testing $H_0 : \theta = 5$ versus $H_1 : \theta \neq 5$, specify the distribution and verify the null hypothesis.

Exercise 3

(3 points) Total precipitation (in mm) has been recorded on a daily basis at the Vancouver airport for many years. The largest of these daily totals for a given years is called the annual maximum; it describes the amount of precipitation on the "wettest" day of the year. You have a data file listing the annual maxima for the last n years. These n annual maxima do not show any trend over time, so a statistical model in which these are modelled as independent and identically distributed random variables seems reasonable. Assume that the exponential distribution provides a reasonable model. So, if X_i denotes the annual maxima for year i, our statistical models is

$X_1, X_2, \ldots, X_n$ is a simple random sample from the population with density function $f(x)$ given by:

$$f(x) = \lambda exp(-\lambda x) \qquad for\ x > 0$$

Our primary interest is in θ, the probability that next year's annual maximum will exceed 100mm.

1. Express θ as a function of λ
2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and its asymptotic distribution

Exercise 4

A company contacts potential customers by phone until the first sale is made. Assume that each call results in a sale with probability p independently of the previous calls.

Let $X_1, \ldots, X_n$ be a random sample from a Geometric distribution:

$$P(X = x) = p(1-p)^{x-1}, \qquad x = 1, 2, \ldots$$

where X represents the number of calls required to obtain the first sale.

The company is interested in the probability that the first three calls are all unsuccessful.

1. **(1 point)** Express this probability as a function of p.
2. **(2 points)** Propose an unbiased estimator of this probability.
3. **(1 point)** Suppose that for $n = 10$ customers the following observations were recorded:

   $$3,\ 4,\ 5,\ 6,\ 1,\ 2,\ 7,\ 3,\ 4,\ 5.$$

   Compute the value of your estimator.

Exercise 5 (6 points)

1. Explain the difference between:
   - a statistic;

- a parameter;
- an estimator.

2. What is the purpose of Fisher Information?

3. Explain briefly why asymptotic results are fundamental in modern statistics.