Exam Statistics

SOLUTIONS

10^{th} July 2026

Solutions

Exercise 1 (3 points)

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables. Let $f_\theta(x)$ and $F_\theta(x)$ be the density function and cumulative distribution function respectively. Let $\hat{\theta}$ be the MLE of θ, θ_0 be the true parameter, $L(\theta)$ be the likelihood function, $logL(\theta)$ be the loglikelihood function, and $I(\theta)$ be the Fisher information matrix.

1. To verify the hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$, which of the following statements is NOT true:

   (a) Wald test requires only the unrestricted model to be estimated.

   (b) Score test requires only the restricted model to be estimated.

   (c) LR test requires both the restricted and unrestricted models to be estimated

   (d) The three statistical tests provide the same values in the observed sample

2. For each of the following statements indicate whether it is TRUE or FALSE:

An estimator $T = T(X)$ is a consistent estimator for θ if the distribution is asymptotically Gaussian	T	F
A statistic $T = T(X)$ is a sufficient statistic for θ if it can be computed without knowing the value of θ	T	F
The likelihood ratio is a random variable	T	F
The following function is a density function f(x)= $\begin{cases} 6x^2 - 1 & if x \in [0,1] \\ 0 & otherwise \end{cases}$	T	F

Exercise 1: SOLUTION

1. The statement that is NOT true is

$$\boxed{(d)}$$

since the Wald, Likelihood Ratio and Score tests are asymptotically equivalent, but they generally produce different values in a finite sample.

2. (a) An estimator is consistent if its distribution is asymptotically Gaussian.

$$\boxed{\text{FALSE}}$$

Asymptotic normality does not imply consistency.

(b) A statistic may be computed without knowing the value of the parameter.

$$\boxed{\text{TRUE}}$$

A statistic is a function of the sample only.

(c) The likelihood ratio is a random variable.

$$\boxed{\text{TRUE}}$$

It depends on the observed sample.

(d)

$$f(x) = \begin{cases} 6x^2 - 1, & x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

is a density function.

$$\boxed{\text{FALSE}}$$

because

$$f(0) = -1 < 0.$$

A density must be non-negative.

Exercise 2

The density is

$$f(x;\theta) = \frac{(\theta-1)\,2^{\theta-1}}{x^\theta} \qquad x > 2 \quad \theta > 1$$

1. Sufficient statistic

The joint distribution is

$$f(x_1, \ldots, x_n; \theta) = \frac{(\theta - 1)^n \, 2^{n(\theta-1)}}{\prod_i x_i^\theta}$$

Therefore the sample enters the likelihood only through

$$\left(\prod X_i\right)$$

or

$$T = \sum_{i=1}^{n} \log X_i.$$

By the Factorization Theorem,

$$\boxed{T = \sum_{i=1}^{n} \log X_i}$$

is sufficient for θ.

2. Maximum Likelihood Estimator

The likelihood is

$$L(\theta) = \frac{(\theta - 1)^n \, 2^{n(\theta-1)}}{\prod_i x_i^\theta}$$

The log-likelihood is

$$\ell(\theta) = n \log(\theta - 1) + n(\theta - 1) \log 2 - \theta \sum_i log(X_i).$$

The score function is

$$U(\theta) = \frac{n}{\theta - 1} + n \log 2 - \sum_i log(X_i).$$

Setting

$$U(\theta) = 0$$

gives

$$\boxed{\hat{\theta}_{MLE} = 1 + \frac{n}{\sum_i log(x_i) - n \log 2}}.$$

3. Method of Moments

First compute

$$\begin{aligned} E(X) &= \int_2^\infty f(x;\theta)dx = \int_2^\infty x\frac{(\theta-1)2^{\theta-1}}{x^\theta}dx = \int_2^\infty (\theta-1)2^{\theta-1}x^{-\theta+1}dx \\ E(X) &= (\theta-1)2^{\theta-1}\Big|_2^\infty \frac{x^{-\theta+2}}{-\theta+2} = 2\frac{\theta-1}{\theta-2} \end{aligned}$$

$$\begin{aligned} \bar{X} &= 2\frac{\theta-1}{\theta-2} \\ \hat{\theta}_{MOM} &= 2\frac{\bar{X}-1}{\bar{X}-2} \end{aligned}$$

4. Fisher Information

The second derivative is

$$\ell''(\theta) = -\frac{n}{(\theta-1)^2}.$$

Hence

$$\boxed{I(\theta) = \frac{n}{(\theta-1)^2}}.$$

5. Asymptotic Distribution

Since

$$I(\theta) = \frac{1}{(\theta-1)^2}$$

per observation,

$$\boxed{\sqrt{n}(\hat{\theta}-\theta) \Rightarrow N\left(0, (\theta-1)^2\right)}.$$

Equivalently,

$$\boxed{\hat{\theta} \approx N\left(\theta, \frac{(\theta-1)^2}{n}\right).}$$

6(a) Likelihood Ratio Test

Given

$$n = 120, \qquad T = 115, \qquad \theta_0 = 5.$$

The MLE is

$$\hat{\theta} = 1 + \frac{120}{115 - 120 \log 2} \simeq 4.77.$$

The statistic is

$$LR = 2\left[\ell(\hat{\theta}) - \ell(\theta_0)\right].$$

Numerically,

$$LR \approx 0.4256.$$

Under H_0, for n large

$$LR \approx \chi_1^2.$$

Therefore, we DON'T reject H_0 at the 1% level.

$$\boxed{\text{Reject } H_0}$$

6(b) Wald Test

$$W = \frac{(\hat{\theta} - 5)^2}{(\hat{\theta} - 1)^2/120} \approx 0.4428.$$

Under H_0, for n large

$$W \approx \chi_1^2.$$

Therefore, we DON'T reject H_0 at the 1% level.

6(c) Score Test

$$U(5) = \frac{120}{4} + 120 \log 2 - 115 = -1.822$$

and

$$I(5) = \frac{120}{16} = 7.5.$$

Therefore

$$S = \frac{U(5)^2}{I(5)} = 0.4428$$

Therefore, we DON'T reject H_0 at the 1% level.

Exercise 3

1. Parameter of Interest

For an exponential random variable,

$$P(X > 100) = e^{-100\lambda}.$$

Thus

$$\boxed{\theta = e^{-100\lambda}}.$$

2. MLE of θ

The MLE of λ is

$$\hat{\lambda} = \frac{1}{\bar{X}}.$$

Using the invariance property,

$$\boxed{\hat{\theta} = e^{-100\hat{\lambda}} = e^{-100/\bar{X}}.}$$

3. Asymptotic Distribution

Since

$$\hat{\lambda} \approx N\left(\lambda, \frac{\lambda^2}{n}\right),$$

and

$$g(\lambda) = e^{-100\lambda},$$

$$g'(\lambda) = -100e^{-100\lambda},$$

the Delta Method yields

$$\boxed{\hat{\theta} \approx N\left(e^{-100\lambda}, \frac{10000\lambda^2 e^{-200\lambda}}{n}\right).}$$

Exercise 4

1. Probability of Interest

The first three calls are unsuccessful if

$$X > 3.$$

Hence

$$\boxed{\theta = P(X > 3) = (1-p)^3.}$$

2. Unbiased Estimator

Define

$$I_i = \mathbf{1}_{\{X_i > 3\}}.$$

Then

$$E(I_i) = P(X_i > 3) = (1-p)^3.$$

Therefore

$$\boxed{\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_i > 3\}}}$$

is unbiased.
Indeed,

$$E(\hat{\theta}) = (1-p)^3 = \theta.$$

3. Numerical Value

The observed sample is

$$3,\ 4,\ 5,\ 6,\ 1,\ 2,\ 7,\ 3,\ 4,\ 5.$$

The indicators are

$$0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1.$$

Therefore

$$\sum I_i = 6.$$

Hence

$$\boxed{\hat{\theta} = \frac{6}{10} = 0.6.}$$

Exercise 5

1. A parameter is a fixed but unknown characteristic of a population.

   A statistic is a function of the sample observations only.

   An estimator is a statistic used to estimate an unknown parameter.

2. Fisher Information measures the amount of information contained in the sample about the unknown parameter. Larger Fisher Information implies more precise estimation and smaller asymptotic variance.

3. Asymptotic results are fundamental because exact finite-sample distributions are often difficult to obtain. They provide approximations used for estimation, confidence intervals and hypothesis testing. Examples include consistency and asymptotic normality of the MLE.