Giulia Pavan

Probability
Problem Set 4 - Solutions

$\hookrightarrow$ **Topics**
Conditional probability and independence, partition, conditional probability, Binomial distribution, Poisson distribution, Uniform distribution.

We are using

- $P(A \cap B) = P(A)P(B) \leftrightarrow A \perp B$
- $P(A) = P(A \cap B) + P(A \cap B^C)$
- $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- $X \sim B(n, p)$ means that $P(X = x) = \binom{n}{x} p^x (1-p)^n \qquad x = 0, 1, ..., n$

Exercise 1

Show that $A \perp B \rightarrow A^C \perp B^C$

Solution
$A^C \perp B^C$ means that

$$P(A^C \cap B^C) = P(A^C)P(B^C)$$

We want to show that from the two sides we get the same results.
Right hand side (using the hypothesis and the De Morgan's law $(A \cup B)^c = A^c \cap B^c$)

$$1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) + P(A)P(B)$$

Left hand side (using the definition of complements and the distributive property)

$$(1 - P(A))(1 - P(B)) = 1 - P(A) - P(B) + P(A)P(B)$$

Exercise 2

Define the events
A = ill
B = smoker
and define the probabilities
$P(B) = 0.4$
$P(A \mid B) = 0.25$
$P(A \mid B^C) = 0.07$
What is the probability of being ill?

Solution

$$P(A) = P(A \cap B) + P(A \cap B^C) = P(A \mid B)P(B) + P(A \mid B^C)P(B^C) =$$
$$= (0.25)(0.4) + (0.07)(0.6) = 0.142$$

What is the probability of being smoker given that you are ill?
Solution

$$P(B \mid A) = \frac{P(A \cap B)}{P(B)} = \frac{(0.25)(0.4)}{0.142} = 0.704$$

Exercise 3
Given a package with three balls, define X=number of broken balls in a package and $p = 0.2$ the probability of a ball to be broken. (We are assuming that the fact that a ball is broken is independent on the state of the other balls). Which is the probability that the number of broken balls is less than one?

$$X \sim B(3, 0.2)$$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \binom{3}{0} 0.2^0(0.8)^3 + \binom{3}{1} 0.2^1(0.8)^2 = \frac{112}{125}$$

Exercise 4
28 people booked a flight. The probability that each passenger is coming at the check-in is 0.7. Which is the probability that more than 25 passenger come at the check-in? (We are assuming that each passenger is independent from the others)

$$X \sim B(28, 0.7)$$

$$P(X \geq 25) = \sum_{x=25}^{28} \binom{28}{x} 0.7^x 0.3^{28-x} =$$

$$= \binom{28}{25} 0.7^{25} 0.3^{3} + \binom{28}{26} 0.7^{26} 0.3^{2} + \binom{28}{27} 0.7^{27} 0.3^{1} + \binom{28}{28} 0.7^{28} 0.3^{0} = 0.0157$$

Exercise 5
Compute the expected value of the r.v. X distributed as a Poisson.
Solution

$$X \sim \text{Poisson}(\lambda)$$
$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$
$$E(X) = \sum_x xP(X = x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

define $y = x - 1$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{y!} =$$

since $\lambda^{y+1} = \lambda^y \lambda$

$$= e^{-\lambda}\lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

because $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Definition

A *density* is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \geq 0$ and $\int_{\mathbb{R}} f = 1$. A random variable X is *absolutely continuous* if there exist a density f such that $P(a < X < b) = \int_a^b f$.

For an absolutely continuous r.v. the expectation value is given by

$$\mathbb{E}(X) = \int_{\mathbb{R}} x f(x) \mathrm{d}x$$

Exercise 6
Compute the mean, the second moment and the variance of a uniform distribution.
Solution
The probability density function of a uniform distribution is:

$$f(x, y) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

Expected value

Since the function is different from zero only in the interval $[a, b]$, we compute the integral within that interval.

$$E(X) = \int_a^b xf(x)\,\mathrm{d}x = \int_a^b x\frac{1}{b-a}\,\mathrm{d}x =$$

$$= \frac{1}{b-a}(\frac{b^2}{2} - \frac{a^2}{2}) = \frac{1}{2(b-a)}(b^2 - a^2) = \frac{a+b}{2}$$

Second moment

$$E(X^2) = \int_a^b x^2 f(x)\,\mathrm{d}x = \int_a^b x^2\frac{1}{b-a}\,\mathrm{d}x =$$

$$= \frac{1}{b-a}(\frac{b^3}{3} - \frac{a^3}{3}) = \frac{1}{3(b-a)}(b^3 - a^3) = \frac{b^3 + a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3(b-a)}$$

Variance

$$Var(X) = E[X^2] - E[X]^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a+b)^2}{4} =$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 3b^2 - 6ab}{12} = \frac{a^2 + b^2 - 2ab}{12} = \frac{(a-b)^2}{12}$$