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Probability
Problem Set 6 - Solutions

$\hookrightarrow$ **Topics**

Expected value, variance, second moment, geometric distribution, Poisson distribution

We are using

- first derivative of power series $\displaystyle\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$
- second derivative of power series $\displaystyle\sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$
- $T \sim Geom(n,p)$ the probability mas function is $P(T=n) = pq^{n-1}$
- $X \sim$ Poisson(λ) the probability mass function is $P(X=x) = \frac{e^{-\lambda}\lambda^k}{K!}$
- Cebicev inequality $P(|V - \mathbb{E}(V)| \geq a) \leq \frac{\mathrm{Var}(V)}{a^2} \qquad a > 0$

Exercise 1

Compute the secnd moment and the variance of the r.v. T ditributed as a geometric distribution

$$\mathbb{E}(T^2) = \sum_n n^2 P(T=n) = \sum_{n=1}^{\infty} n^2 p q^{n-1} =$$

$$= p(\sum_{n=1}^{\infty} n^2 q^{n-1}) = p\left(\sum_{n=1}^{\infty}(n^2-n)q^{n-1} + \underbrace{\sum_{n=1}^{\infty} n q^{n-1}}_{\text{first derivative of power series}}\right) =$$

$$= p\left(\sum_{n=1}^{\infty} n(n-1)q^{n-1} + \frac{1}{(1-q)^2}\right) = p\left(q \underbrace{\sum_{n=1}^{\infty} n(n-1)q^{n-2}}_{\text{second derivative of power series}} + \frac{1}{(1-q)^2}\right) =$$

$$= p\left(q\frac{2}{(1-q)^3} + \frac{1}{(1-q)^2}\right) = p\left(q\frac{2}{p^3} + \frac{1}{p^2}\right) = \frac{2q+p}{p^2} = \frac{q+1}{p^2}.$$

Variance

$$\mathrm{Var}(T) = \mathbb{E}(T^2) - (\mathbb{E}(T))^2 = \frac{q+1}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

Exercise 2

Compute the second moment and the variance of the r.v. X distributed as a Poisson.

Solution

$$X \sim \mathrm{Poisson}(\lambda)$$

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\mathbb{E}(X^2) = \sum_k k^2 P(X = k) = \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} \frac{k\lambda^k}{(k-1)!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!}$$

Consider only the summation

$$\sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} = \sum_{k=1}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!} + \frac{\lambda^{k-1}}{(k-1)!} =$$

$$\sum_{k=2}^{\infty} \frac{\lambda^{k-1}}{(k-2)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

define $k - 1 = m$

$$\lambda\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}$$

define $k - 2 = j$

$$= \lambda\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + e^{\lambda} = e^{\lambda} + \lambda e^{\lambda} = e^{\lambda}[1+\lambda]$$

because $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Therefore

$$\mathbb{E}(X^2) = \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{k\lambda^k}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda}[1+\lambda] = \lambda(1+\lambda) = \lambda + \lambda^2$$

The variance is

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Exercise 3

U is a random variable with a density given by

$$f_U(x) = 2\frac{\log x}{x} 1_{[1,c]}(x) \qquad c > 1$$

i) Calculate c. ii) Calculate $\mathbb{E}(U^2)$ e $P(0 < U < 1)$.

Solution

- Remark! $f_U \geq 0$
- $D(f(x)^2) = 2 \cdot f(x) \cdot f'(x)$

$$1 = \int_{\mathbb{R}} f_U(x)\mathrm{d}x = \int_{\mathbb{R}} 2\frac{\log x}{x} 1_{[1,c]}(x)\mathrm{d}x = \int_1^c 2\frac{\log x}{x}\mathrm{d}x =$$
$$= [(\log x)^2]_1^c = (\log c)^2 - (\log 1)^2 = (\log c)^2$$

$$(\log c)^2 = 1 \qquad \Rightarrow \qquad \log c = \pm 1 \quad c = e, \ \frac{1}{e}$$

This implies $c = e$ (since $c \geq 1$).
ii) Integrating by parts we get

$$\mathbb{E}(U^2) = \int_{\mathbb{R}} x^2 f_U(x)\mathrm{d}x = \int_1^e x^2 \cdot 2\frac{\log x}{x}\mathrm{d}x = \int_1^e 2x \cdot \log x \,\mathrm{d}x =$$

$$= \left[x^2\left(\log x - \frac{1}{2}\right)\right]_1^e = e^2\left(\log e - \frac{1}{2}\right) - 1^2\left(\log 1 - \frac{1}{2}\right) = \frac{1}{2}(e^2 + 1)$$

Finally

$$P(0 < U < 1) = \int_0^1 f_U(x)\mathrm{d}x = \int_0^1 0 \,\mathrm{d}x = 0$$

Exercise 4

$V \sim$ Poisson(2). Order the following three numbers from the smallest to the biggest.

$$\frac{2}{9} \qquad 2 \cdot F_V(0) \qquad P(|V-2| \geq 3)$$

Solution

If $X \sim$ Poisson(λ) then $\mathrm{Var}(X) = \mathbb{E}(X) = \lambda$.
Cebicev inequality says that

$$P(|V - \mathbb{E}(V)| \geq a) \leq \frac{\mathrm{Var}(V)}{a^2} \qquad\qquad a > 0$$

Therefore

$$P(|V-2| \geq 3) \leq \frac{2}{9}$$

On the other side

$$2 \cdot F_V(0) = 2 \cdot P(V \leq 0) = 2 \cdot P(V = 0) = 2 \cdot e^{-2}\frac{2^0}{0!} = \frac{2}{e^2} > \frac{2}{9}.$$

Finally

$$P(|V-2| \geq 3) \leq \frac{2}{9} \leq 2 \cdot F_V(0)$$