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Probability
Problem Set 7 - Solutions

$\hookrightarrow$ **Topics**

Exercise 1.

Define three random variables:
$X \sim N(0,1)$
$P(Z=1) = P(Z=-1) = \frac{1}{2}$, and $X \perp Z$ $Y = ZX$ Proposition: $Y \sim N(0,1)$, $X + Y \nsim N(\mu, \sigma^2)$

Show that $Y \sim N(0,1)$:

$$P(Y \in A) = P(ZX \in A) = P(Z = 1 \cap X \in A) \dot{\cup} P(Z = -1 \cap -X \in A) =$$

$$= P(Z=1)P(X \in A) + P(Z=-1)P(-x \in A) = \frac{1}{2}P(X \in A) + \frac{1}{2}P(-X \in A) =$$
$$= P(X \in A)$$

Notice that since Normal distribution is symmetric, then $X \sim N(0,1) \Rightarrow -X \sim N(0,1)$

$$P(X + Y = 0) = P(X + ZX = 0) = P(Z = -1) = \frac{1}{2}$$

Exercise 2.

(X,Y) is a random vector with uniform density on $A = \{(x,y) \in R^2 | x^2 + y^2 \leq 1\}$.

Let $U := X + Y$.

i) Are X and Y independent?

ii) Calculate $F_U(-2)$, $F_U(0)$, $F_U(2)$.

iii) Let f_X be the density of X: where does f_X have its maximum?

i) X and Y are not independent because the set A is not a Carthesian product, which is a necessary (but not sufficient) condition for independence.

ii) $F_U(-2) = P(X + Y \leq -2) = 0$; $F_U(0) = P(X + Y \leq 0) = \frac{1}{2}$ and $F_U(2) = P(X + Y \leq 2) = 1$

iii) it reaches its maximum at $x = 0$.

Exercise 3.

Calculate the characteristic function of a random variable distributed as a binomial.

$$B \sim B(n,p)$$

$$\varphi_X(t) = \mathbb{E}(e^{itX}) = \sum_{k=0}^{n} e^{itk} \binom{n}{k} p^k q^{n-k} =$$

$$\sum_{k=0}^{n} \binom{n}{k} (e^{it}p)^k q^{n-k} = (e^{it}p + q)^n$$

Because of the binomial theorem.

Exercise 3.

Prove that

$$\mathbb{E}(X|Y) = \mathbb{E}(X) \Rightarrow \mathrm{Cov}(X, Y) = 0$$

but not viceversa

Proof $\Rightarrow$

Given the definition of covariance: $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

$$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY|Y) = \mathbb{E}(Y\mathbb{E}(X|Y)) =$$

Because of the law of iterated expectation

$$\mathbb{E}(Y(\mathbb{E}(X)) = \mathbb{E}(Y)\mathbb{E}(X)$$

Using the hypothesis we showed the thesis.

Proof $\not\Leftarrow$

Use a counterexample.

Consider: $P(X = i) = \frac{1}{3}$, $i = -1, 0, 1$ and $X^3 = X$

define: $Y = X^2$

$$\left.\begin{array}{l} \mathbb{E}(XY) = \mathbb{E}(X^3) = \mathbb{E}(X) = 0 \\ \mathbb{E}(X)\mathbb{E}(Y) = 0 \end{array}\right\} \mathrm{Cov}(X, Y) = 0$$

$$\mathbb{E}(Y|X) = \mathbb{E}(X^2|X) = X^2 \neq \mathbb{E}(Y) = \frac{2}{3}$$

Namely, $\mathbb{E}(Y) = \mathbb{E}(X^2) = 0P(X^2 = 0) + 1P(X^2 = 1) = 1(P(X = 1) + P(X = -1)) = 1(\frac{1}{3} + \frac{1}{3}) = \frac{2}{3}$