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Mathematics – Teacher: Paolo Gibilisco

(Simulation 1 of) Written Examination

Rules: you cannot use any textbook, lecture notes, neither any electronic device (very welcome a traditional watch). Write your solutions on the enclosed sheets. You will receive additional sheets for the preliminary drafts. The solutions of the exercises must be detailed. One of the exercises requires the proof of a theorem. For the quizzes you have simply to indicate your choice in the True-False alternative. You have **two hours and half** to finish your work. An **identity card** or a passport is needed to participate.

Quiz 1. *Correct answer: points 2. Wrong answer: points -1. No answer: points 0.*
If $X \sim \mathcal{N}(0,1)$ then the characteristic functions $\varphi_X(x) = e^{-\frac{x^2}{2}}$ is a solution of the Cauchy problem

$$\begin{cases} y' = -xy \quad x \in \mathbb{R} \\ y(0) = 0 \end{cases}$$

TRUE FALSE

Quiz 2.
The matrix

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is not orthogonal.

TRUE FALSE

Quiz 3.
Suppose that V_1, V_2, V_3 are vector spaces and that $T_1 : V_1 \to V_2$, $T_2 : V_2 \to V_3$ are some functions. If $T_2 \circ T_1 : V_1 \to V_3$ is a linear transformation then T_1, T_2 are linear transformations too.

TRUE FALSE

Quiz 4.
Let A, B be square matrices. Define the *commutator* as $[A,B] := AB - BA$. Then

$$\mathrm{Tr}([A,B]) = 0$$

TRUE FALSE

Quiz 5.
Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Let $a < b$ and suppose that $f'(a) < 0$, $f'(b) > 0$. This implies that (for f) there exists a unique stationary point $x_0 \in (a,b)$.

TRUE FALSE

Quiz 6.
The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = -3x^4 + 5x^3 + x^2 - x - 15$$

has a global maximum on $\mathbb{R}$.

TRUE FALSE

Quiz 7.
Let $X \sim B(n, p)$. Suppose that $P(X = 0) = 1$. This implies that $P(X = n) = 0$.

TRUE FALSE

Exercise 1. *6 points*

Calculate

$$\int\int_C (2xy - x)\,\mathrm{d}x\mathrm{d}y$$

where

$$C = \{(x, y) \in \mathbb{R}^2 \,|\, y < -x^2 + 1,\ y > x^2 - 1\}$$

Exercise 2. *6 points*

Let

$$A = \begin{pmatrix} 9 & 6 & 3 \\ 6 & 13 & 8 \\ 3 & 8 & 6 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

i) Find the Cholesky decomposition $A = LL^t$.
ii) For L given by the point *i)* solve the system $LX = b$.

Exercise 3. *6 points*

Find the stationary points of the following function and discuss the behavior of the function in those points.

$$g(x, y) = e^{x^2} + xy - y^2 + 3.$$

Exercise 4. *6 points*

Suppose that you flip a fair coin which has "0" and "1" on its faces and that you roll, independently, a fair die. Let us denote by X the result of the coin and by Y the result of the die. Let $Z = XY$.
a) Which is the distribution of Z?
b) Calculate $\mathbb{E}(Z)$.
c) Calculate $\mathrm{Var}(Z)$.

Exercise 5. *6 points*

Prove the Central Limit Theorem.