# Notes on Expected Revenue from Auctions

## Economics 100C, Professor Bergstrom

These notes spell out some of the mathematical details about first and second price sealed bid auctions that were discussed in Thursday's lecture. You can find an even more detailed discussion in Steve Matthews' Technical Primer, which is linked to the class website. Although there is quite a bit of detail, you will see that the only mathematics used is simple probability theory, a little algebra and some easy calculus of a single variable–differentiating and integrating polynomial functions.

The analysis for 2 bidders is fairly simple and I urge you to try to follow the argument here for this case. The $n$ bidder case requires some more calculation, but uses the same road map to its results. Even if you don't follow the entire argument for the $n$ bidder case, it is valuable to compare these results with the two bidder case and to understand what happens as you add more bidders.

## Private Value Auctions

Suppose that a supplier has a single object for sale and that there are $n$ possible buyers. The value of the object to person $i$ is some number $v_i$ that is known to $i$ but not known to anyone else. If Person $i$ gets the object for price $p$, his profit will be $v_i - p$. Let us suppose that everybody except person $i$ thinks that $i$'s value for the object is a random variable drawn from the same probability distribution. If random variable is continuous, we let $F(x)$ denote the probability that this random variable is less than or equal to $x$ and $f(x) = F'(x)$ be the corresponding density function.

We will often work with the example of a uniform distribution on the interval $[0, 100]$. Sometimes it is convenient to think of a discrete version of this random variable in which the random variable takes on only integer values and where the probability the that $x = n$ for any integer $n$ between 1 and 100 is $f(n) = 1/100$. Then the probability that $x \leq n$ must be $F(n) = n/100$. Here we will work with the continuous uniform distribution on $[0, 100]$. In this case, for any real number $x$ between 0 and 100, we have $F(x) = x/100$ and $f(x) = F'(x) = 1/100$.

# How to bid in a second-price sealed bid auction

In a second-price sealed bid auction, the object is sold to the high bidder. The high bidder does not pay his own bid, but instead pays the amount bid by the second-highest bidder. If your bid is not the high bid, you don't get the object and you don't have to pay anything. In this auction, your own bid does not affect the amount that you would pay if you got the object. It only determines whether you get the object. A remarkable feature of the second-price auction is that no matter what you believe about other people's strategies, the best thing for you to do is to bid your true value. Why is this so? Whatever happens in the auction, there will be some bid $z$ that is the highest bid made by the other guys. If your own value is $v$, then whatever the value of $z$ is, your profit will be $v - z$ if your bid is greater than $z$ (because you get the object and pay $z$ for it) and it will be 0 if you bid less than $z$ (because in this case you don't get the object and don't pay for it). What difference would it make if you tried to "overbid" by bidding $b > v$? If $z < v$, it wouldn't matter at all, because you would win the object and pay $z$ just as you would if you bid your true value $v$. But what if $b > z > v$? In this case, you would not win the object if you bid your true value $v$, but you will win the object with your bid of $b$. But if $b > z > v$, then your profit is $v - z < 0$. You will be paying more for the object than it is worth to you and will be worse off than if you had bid the truth and not won the object. This means that you can never make yourself better off and you may make yourself worse off by overbidding than by bidding your true values. What difference would it make if you tried "underbidding", by bidding less than your true valuation? If your bid is $b < v$ and if $z < b$, it will make no difference that you underbid, since you will still win the object and you will still pay $z$ for it. But if it happens that $b < z < v$, then with a bid of $b$, you will not win the object and will get 0 profit. If you had bid your true value $v$, you would have won the object and had a profit of $v - z > 0$. So you cannot help yourself and you may hurt yourself by bidding less than your true value. It follows that no matter what the other bidders are doing, the best thing for you to do is to bid your true value. Bidding anything else cannot help you and may hurt you. A strategy for which this is the case is called a *weakly dominant strategy*. Thus bidding your true value in a second-price sealed bid auction is a weakly dominant strategy.

## How to bid in a first-price sealed bid auction

In a first-price sealed bid auction, the object is sold to the high bidder at the high bidder's bid. In this case, it is easy to see that there will not be a weakly dominant strategy. (A strategy is weakly dominant if one could not do better even if one knew what the other players are doing. In the case of a first-price auction, the best thing to do depends on what the other bidders are doing. For example if the value to you is $100 and the highest bid by anyone else is $40, you are best off bidding just a tiny bit more than $40, but if the highest bid by anyone else is $50, you are best off bidding just a bit more than $50. So if there is no weakly dominant strategy, what do you do when you don't know what the other bidders are doing? A reasonable approach is to maximize the expected value of your profit. If you bid $b$ and your value is $v$, your expected profit will $(v-b)P(b)$ where $P(b)$ is the probability that you get the object if you bid $b$. If higher bids are more likely to win the object, then $P(b)$ will be an increasing function of $b$ while $v - b$ will be a decreasing function of $b$. You are going to have to make a tradeoff. Let's see how this would work if there are only two bidders and each of them believes that the other's value is a random variable drawn from the uniform distribution on the interval $[0, 100]$. You realize that it would be foolish to bid your true value, since in this case your profit is 0 whether you win the object or not. To get started, assume that the other guy will bid some fraction $c$ of his true value.

Then if you bid $b$, the probability that you will win the object will be the probability that $cx < b$ where $x$ is the other bidder's value. This is the same as the probability that $x < b/c$ where $x$ is the other guy's value. Since have assumed that the other guy's value is a random variable from the uniform distribution on $[0, 100]$, it follows that the other guy's value is less than $b/c$ is $F(b/c) = \frac{b}{100c}$. So we have $P(b) = \frac{b}{100c}$ and expected profits of

$$(v - b)\frac{b}{100c} = \frac{1}{100c}\left(vb - b^2\right).$$

You want to choose $b$ to maximize this expected profit. The first order calculus condition for this maximization is

$$\frac{d}{db}\frac{1}{100c}\left(vb - b^2\right) = \frac{1}{100c}(v - 2b) = 0.$$

(You should check the second order condition to see that this is really a maximum.) The solution of this equation is $v = 2b$, or equivalently, $b = \frac{v}{2}$.

So if you believe that the other guy is going to bid an fraction $c$ of his value, the best thing for you to do is to bid $1/2$ of your value. If he believes the same thing about you, the best thing for him is to bid $1/2$ of his value. So in the case of two bidders with values uniformly distributed on an interval, there is an equilibrium in which each bids half of his value. We note that the object would then always be sold to the bidder with the higher buyer value and it would be sold at a price equal to half of that person's value.

Let's work out the case for $n$ bidders. Suppose that you bid $b$. What is the probability that your bid is the highest one? It is the probability that all $n-1$ of the other guys bid less than $b$. Suppose that each of them bids some fraction $c$ of his true value. Then the probability that any one of them bids less than $b$ is $F(b/c) = \frac{b}{100c}$ and the probability that all of them bid less than $b$ is

$$F(b/c)^{n-1} = \left(\frac{b}{100c}\right)^{n-1} = \frac{b^{n-1}}{(100c)^{n-1}}.$$

Then if you bid $b$, your expected profit is

$$(v-b)\frac{b^{n-1}}{(100c)^{n-1}} = \frac{1}{(100c)^{n-1}}\left(vb^{n-1} - b^n\right).$$

To maximize your expected profit, you would choose $b$ so that

$$\frac{d}{db}\frac{1}{(100c)^{n-1}}\left(vb^{n-1} - b^n\right) = \frac{1}{(100c)^{n-1}}\left((n-1)b^{n-2}v - nb^{n-1}\right) = 0.$$

This is equivalent to

$$(n-1)b^{n-2}v - nb^{n-1} = 0,$$

which is equivalent to

$$b = \frac{n-1}{n}v.$$

Thus we see, that as the number of bidders gets larger, each bidder bids closer to his actual value. If there are 3 bidders, then in equilibrium, each bids $2/3$ of his value. If there are 4 bidders, each bids $3/4$ of his value. If there are 10 bidders, each bids $9/10$ of his value and so on.

## Expected Revenue

If he doesn't know the values of the bidders, a seller will not know in advance whether one type of auction will give him more revenue than another. His

return from either type of auction is a random variable. But given his beliefs about the probability distribution of bidders' values, he can calculate his expected revenue from any type of auction.

## Expected Revenue from First-Price Sealed Bid Auction

We will work this out first for the case of 2 bidders where the values of bidders are independent draws from a uniform distribution on the interval $[0, 100]$.

From our earlier discussion, we have learned that in equilibrium, each bidder will bid half of his value. Thus, the price at which the object will be sold is $1/2$ of the higher of the two bidders' values. To find the probability distribution of the seller's revenue, we first want to find the probability distribution of the higher of the two bidder's values. Let us find the probability $g(x, 2)$ that the highest of 2 bids is $x$. Now $x$ can be the highest bid in two possible ways. One way is that bidder 1 has value $x$ and bidder 2 has value less than or equal to $x$. The probability of this event is $f(x)F(x)$. Since $f(x) = \frac{1}{100}$ and $F(x) = \frac{x}{100}$, the probability of this outcome is $\frac{1}{100}\frac{x}{100}$. The other way in which $x$ can be the highest bid is that bidder 2 has value $x$ and bidder 1 has value less than or equal to $x$. This also happens with probability $\frac{1}{100}\frac{x}{100}$. Therefore the probability that $x$ is the highest value from two bidders is

$$g(x, 2) = \left(\frac{2}{100}\right)\left(\frac{x}{100}\right).$$

We can calculate the expected revenue of the seller as follows. Recall that with two bidders, each bidder bids half of his value. So the expected revenue of the seller must be $1/2$ of the expected value of the random variable which is the highest value bidder's value. Since $g(x, 2)$ is the probability that the highest value is $x$, the expected value of the highest value is

$$\int_0^{100} xg(x, 2)dx = \int_0^{100} x\left(\frac{2}{100}\right)\left(\frac{x}{100}\right)dx = \frac{2}{100^2}\int_0^{100} x^2 dx.$$

Now

$$\int_0^{100} x^2 dx = \frac{1}{3}x^3|_0^{100} = \frac{100^3}{3}.$$

Therefore the expected value of the highest bidder value is

$$\int_0^{100} xg(x, 2) = \frac{2}{100^2}\left(\frac{100^3}{3}\right) = \frac{2}{3}100.$$

Recall that in equilibrium, bidders bid only half their values, so the expected revenue of the seller is $1/2$ of the expected value of the higher bidder value. Hence the expected revenue of the seller is just

$$\frac{1}{3}100.$$

What if there are $n$ bidders? What is the probability $g(x, n)$ that the high bid is $x$ if there are $n$ bidders? Then $x$ could be the highest buyer value in $n$ different ways, one way for each person who could be high bidder. Each of these ways has probability

$$\left(\frac{1}{100}\right)\left(\frac{x}{100}\right)^{n-1}$$

and so it must be that

$$g(x, n) = n\left(\frac{1}{100}\right)\left(\frac{x}{100}\right)^{n-1}.$$

The expected value of the high bid will be the integral

$$\int_0^{100} xg(x, n)dx = \int_0^{100} xn\left(\frac{1}{100}\right)\left(\frac{x}{100}\right)^{n-1}dx = n\left(\frac{1}{100^n}\right)\int_0^{100} x^n dx.$$

Now, since

$$\int_0^{100} x^n dx = \frac{1}{n+1}x^{n+1}\big|_0^{100} = \frac{1}{n+1}100^{n+1},$$

we have

$$\int_0^{100} xg(x, n)dx = \frac{n}{n+1}100.$$

Recall that that when there are $n$ bidders, each bids the fraction $\frac{n-1}{n}$ of his value. Thus where there are $n$ bidders, the expected value of the high bidder's bid is $\frac{n-1}{n}$ times the expected value of the highest value. Thus the seller's expected revenue must be

$$\frac{n-1}{n}\int_0^{100} xg(x, n)dx = \left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right)100 = \left(\frac{n-1}{n+1}\right)100.$$

This is consistent with our previous result for 2 bidders, where expected revenue was $(1/3)100$ . Now we see that if there are 3 bidders, expected revenue will be $(1/2)100$, if there are 4 bidders, expected revenue will be $(3/5)100$ and if there are 10 bidders, expected revenue will be $(9/11)100$.

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**Expected Revenue from Second-price sealed bid auction**

Recall that in a sealed bid second-price auction, the best strategy for any player is to bid her true value. Thus the expected revenue of the seller will be the expected value of the second highest value. Again, we will work this out first for the case of 2 bidders where the values of bidders are independent draws from a uniform distribution on the interval $[0, 100]$.

Let $h(x, 2)$ be the probability that the second highest bid is $x$. This can happen in two different ways. Bidder 1 can have a value of $x$, while the value of the other bidder is greater than or equal to $x$. The probability that a bidder has a value greater than or equal to $x$ is $1 - F(x)$ where $F(x)$ is the probability that a bidder has value less than or equal to $x$. Therefore the probability that bidder 1 has value $x$ and bidder 2 has value at least $x$ is

$$f(x)(1 - F(x)) = \left(\frac{1}{100}\right)\left(1 - \frac{x}{100}\right).$$

The other way in which the second highest bid can be $x$ is if bidder 2 has value $x$ and bidder 1 has value at least $x$. This also happens with probability $f(x)(1 - F(x))$ and so

$$h(x, 2) = 2f(x)(1 - F(x)) = \left(\frac{2}{100}\right)\left(1 - \frac{x}{100}\right).$$

The expected value of $h(x, 2)$ is

$$\int_0^{100} xh(x, 2)dx = \int_0^{100} x\left(\frac{2}{100}\right)\left(1 - \frac{x}{100}\right)dx = \frac{2}{100}\int_0^{100}\left(x - \frac{x^2}{100}\right)dx.$$

Now

$$\int_0^{100}\left(x - \frac{x^2}{100}\right)dx = \frac{x^2}{2} - \frac{x^3}{300}\Big|_0^{100} = \left(\frac{1}{6}\right)100^2.$$

Substituting into our previous expression, we see that

$$\int_0^{100} xh(x, 2)dx = \frac{1}{3}100.$$

Notice that this is exactly the same as the expected revenue of the seller from a first-price auction in which both bidders use equilibrium strategies. Saying this in another way, when there are two bidders, the expected value of the second highest buyer value is equal to half of the expected value of the highest buyer value.

What if there are $n$ bidders? Calculating the expected revenue is a little more complicated now. If person 1 is the second highest bidder, then one of the $n-1$ other bidders must bid at least as much as he does and all of the $n-2$ other bidders have to bid no more than he does. The probability of this happening is $f(x)(n-1)(1-F(x))F(x)^{n-2}$. Since there are $n$ different people who could be the second highest bidder, the probability that the second highest bid is $x$ must be

$$h(x,n) = n(n-1)f(x)(1-F(x))F(x)^{n-2}.$$

Now the expected value of $h(x,n)$ will be

$$\int_0^{100} xh(x,n)dx = n(n-1)\int_0^{100} x\left(\frac{1}{100}\right)\left(1-\frac{x}{100}\right)\left(\frac{x}{100}\right)^{n-2}dx.$$

Simplifying this expression, we have

$$\int_0^{100} xh(x,n)dx = \left(\frac{n(n-1)}{100^{n-1}}\right)\int_0^{100}\left(x^{n-1}-\frac{x^n}{100}\right)dx.$$

Now

$$\int_0^{100}\left(x^{n-1}-\frac{x^n}{100}\right)dx = 100^n\left(\frac{1}{n}-\frac{1}{n+1}\right) = 100^n\left(\frac{1}{n(n+1)}\right).$$

Substituting into the previous expression, we find that the expected value of the second highest bidder's value is

$$\int_0^{100} xh(x,n)dx = \frac{n-1}{n+1}.$$

Since in equilibrium for the second-bidder auction, everyone bids his value, this is also the expected revenue of the seller. Note that this is exactly the same as the expected revenue of a seller in a first-price sealed bid auction.

## The Revenue Equivalence Theorem

Most people are very surprised to learn that in the examples we looked at, if bidders are in equilibrium, the revenue from a first-price auction is exactly the same as that from the second-price auction. Is this just a fluke? Would the result disappear if the distributions of values were not uniform on some

interval? It turns out that this result is a special case of a much more general theorem about private value auctions. The equilibrium expected revenue from first price and second price auctions would be the same so long as the distributions of values are continuous and independent between individuals. Even more remarkably, revenue from many other kinds of auctions would be the same as that from the first and second price auctions, so long as distributions of values are independent and the auctions have the following features. 1) The good always goes to the person with highest buyer value. 2) The person with the lowest buyer value is guaranteed to make zero profits in the auction. This result is known as the "revenue equivalence theorem".